On Approximating the Covering Radius and Finding Dense Lattice Subspaces

Daniel Dadush

Centrum Wiskunde & Informatica (CWI)

ICERM April 2018

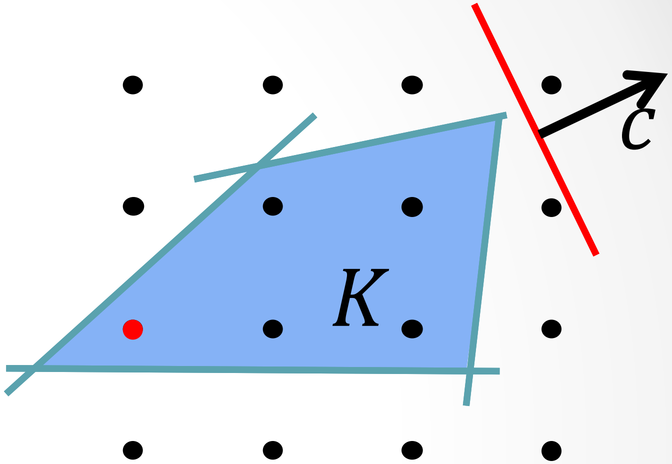
Outline

1. Integer Programming and the Kannan-Lovász (KL) Conjecture.
2. ℓ_2 KL Conjecture & the Reverse Minkowski Conjecture.
3. Finding dense lattice subspaces.

Integer Programming (IP)

$$\min cx$$
$$\text{s.t. } Ax \le b$$
$$x \in \mathbb{Z}^n$$

n variables, m constraints



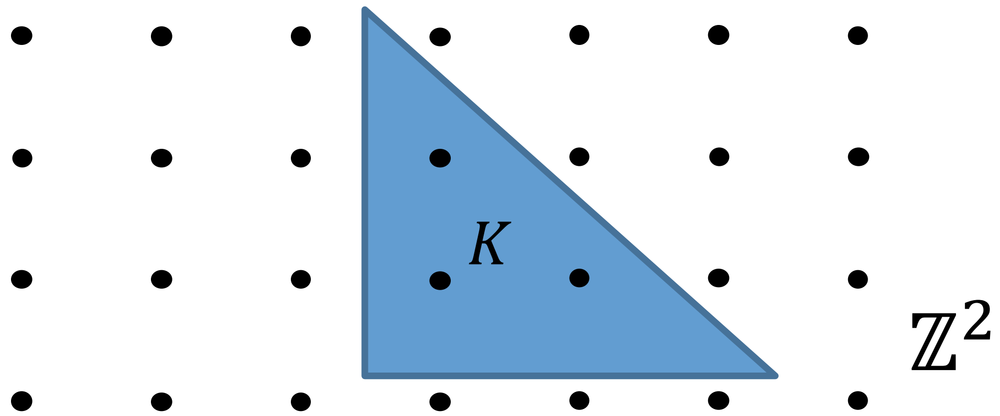
Open Question: Is there a $2^{O(n)}$ time algorithm?

First result: $2^{O(n^2)}$ [Lenstra `83]

Best known complexity: $n^{O(n)}$ [Kannan `87]

Main Dichotomy

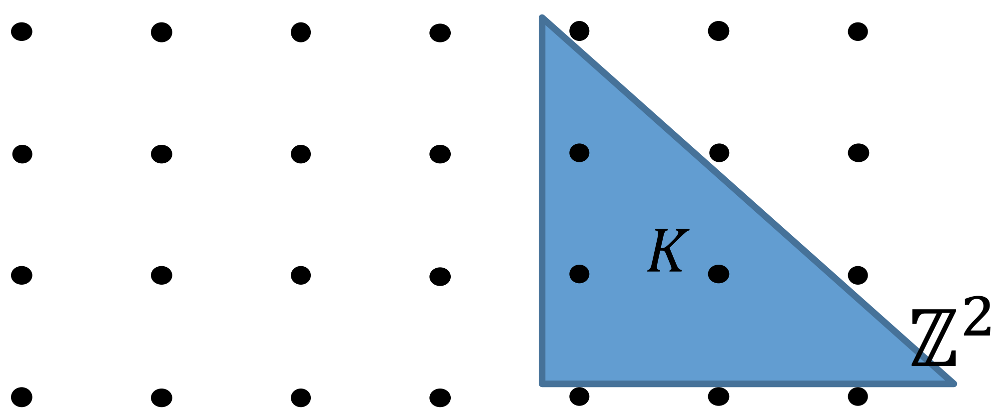
$\mu(K, \mathbb{Z}^n) :=$ **smallest** scaling s such that **every shift** $sK + t$ contains an integer point.



Either **covering radius** $\mu(K, \mathbb{Z}^n) \leq 1$.

Main Dichotomy

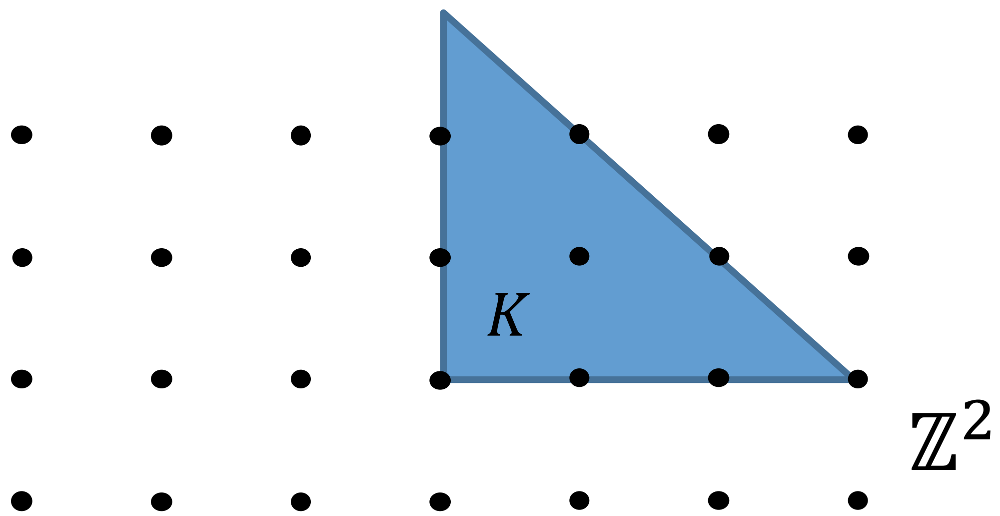
$\mu(K, \mathbb{Z}^n)$:= **smallest** scaling s such that **every shift** $sK + t$ contains an integer point.



Either **covering radius** $\mu(K, \mathbb{Z}^n) \leq 1$.

Main Dichotomy

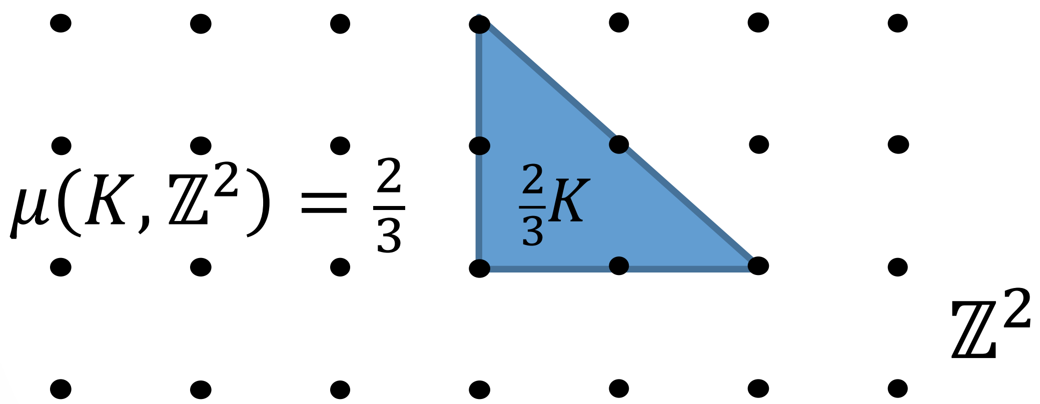
$\mu(K, \mathbb{Z}^n)$:= **smallest** scaling s such that **every shift** $sK + t$ contains an integer point.



Either **covering radius** $\mu(K, \mathbb{Z}^n) \leq 1$.

Main Dichotomy

$\mu(K, \mathbb{Z}^n)$:= **smallest** scaling s such that **every shift** $sK + t$ contains an integer point.

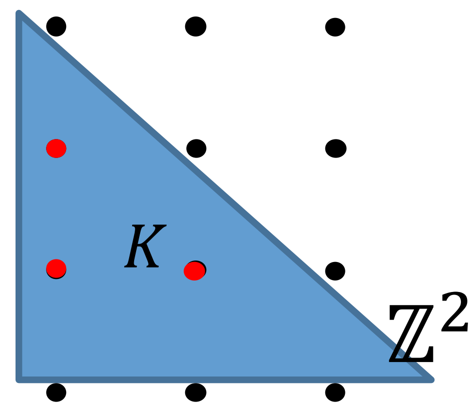


Either **covering radius** $\mu(K, \mathbb{Z}^n) \leq 1$.

Main Dichotomy

Can find integer point in

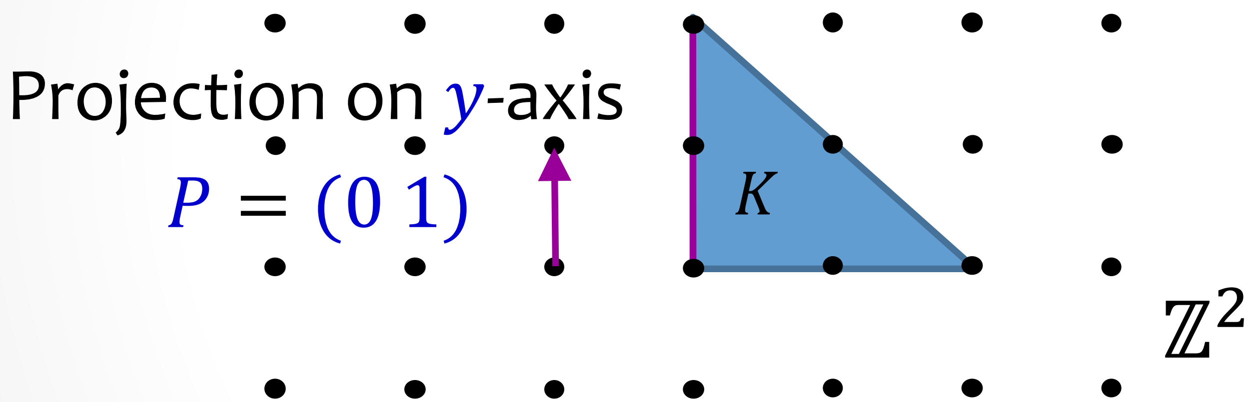
$2^{O(n)}$ time [D. 12]



Either **covering radius** $\mu(K, \mathbb{Z}^n) \leq 1$.

Main Dichotomy

Or K is "flat":

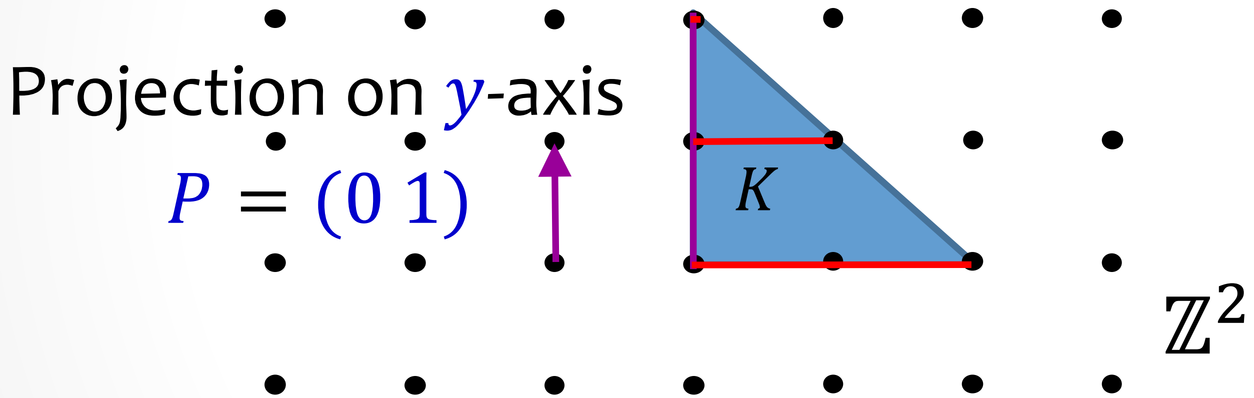


There exists rank $k \geq 1$ integer projection $P \in \mathbb{Z}^{n \times k}$

such $\mathrm{vol}_k(PK)^{\frac{1}{k}}$ is **small**.

Main Dichotomy

Or K is "flat":



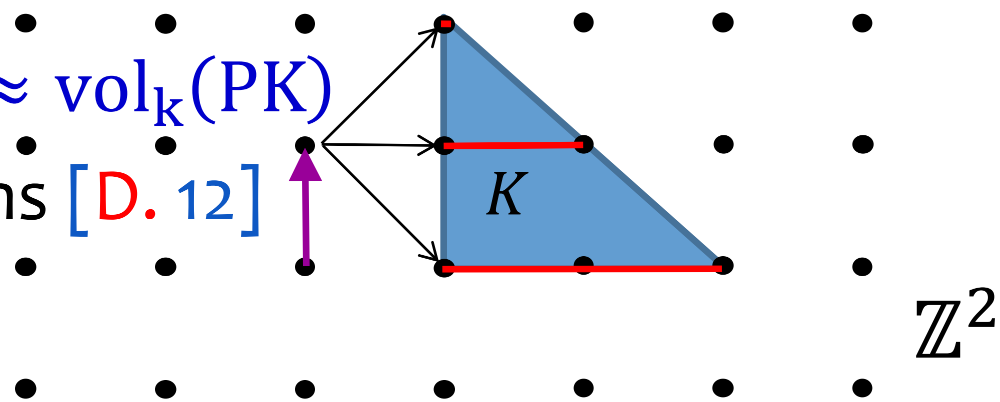
There exists rank $k \geq 1$ integer projection $P \in \mathbb{Z}^{n \times k}$

such $\mathrm{vol}_k(PK)^{\frac{1}{k}}$ is **small**.

Main Dichotomy

Or K is "flat":

Recurse on $\approx \mathrm{vol}_\mathrm{k}(\mathrm{PK})$ subproblems [D. 12]



There exists rank $k \geq 1$ integer projection $P \in \mathbb{Z}^{n \times k}$ such $\mathrm{vol}_k(PK)^{\frac{1}{k}}$ is **small**.

Duality Relation

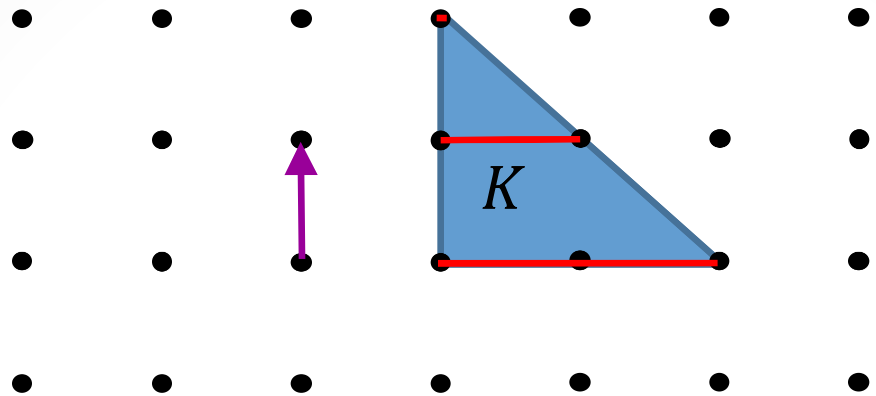
$$1 \leq \mu(K, \mathbb{Z}^n) \min_{\substack{P \in \mathbb{Z}^{k \times n} \\ rk(P) = k \geq 1}} \mathrm{vol}_k(PK)^{\frac{1}{k}} \leq ?$$

"Easy" side

"Hard" side

Either covering radius $\mu(K, \mathbb{Z}^n)$ is small or K is "flat".

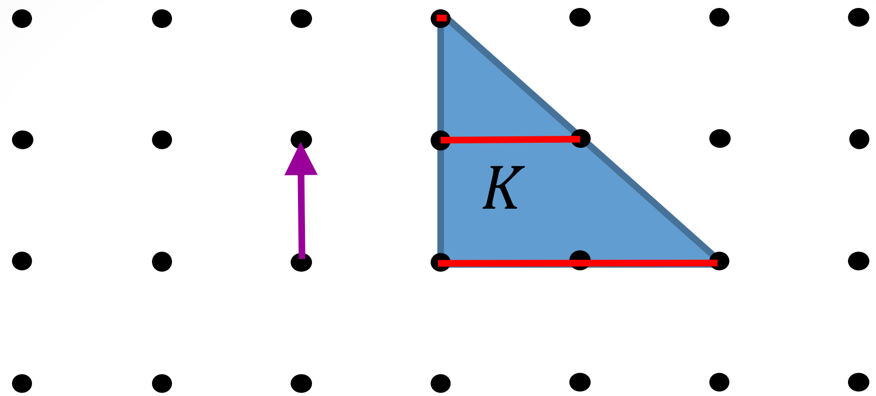
Khinchine Flatness Theorem



$$1 \le \mu(K, \mathbb{Z}^n) \min_{\substack{P \in \mathbb{Z}^{1 \times n} \\ rk(P)=1}} \mathrm{vol}_1(PK) \le \tilde{O}(n^{4/3})$$

[Khinchine `48, Babai `86, Hastad `86, Lenstra-Lagarias-Schnorr `87, Kannan-Lovasz `88, Banaszczyk `93-96, Banaszczyk-Litvak-Pajor-Szarek `99, Rudelson `00]

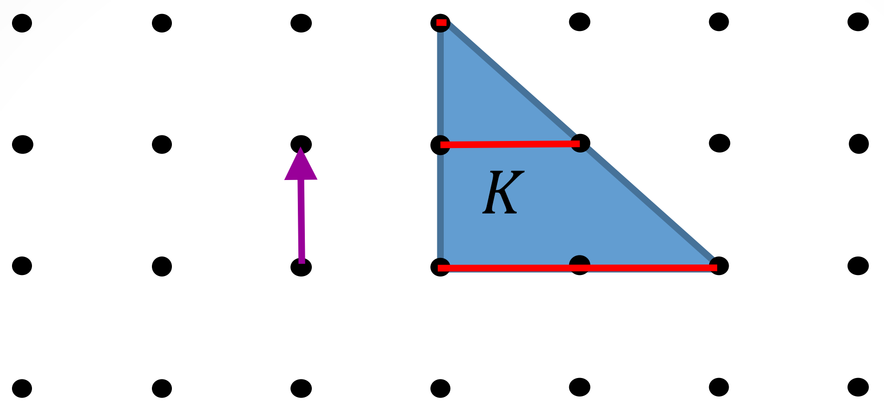
Kannan-Lovász Flatness Theorem



$$1 \leq \mu(K, \mathbb{Z}^n) \min_{\substack{P \in \mathbb{Z}^{k \times n} \\ rk(P) = k \geq 1}} \mathrm{vol}_k(PK)^{\frac{1}{k}} \leq n$$

[Kannan `87, Kannan-Lovász `88]

Kannan-Lovász (KL) Conjecture



$$1 \leq \mu(K, \mathbb{Z}^n) \min_{\substack{P \in \mathbb{Z}^{k \times n} \\ rk(P) = k \geq 1}} \mathrm{vol}_k(PK)^{\frac{1}{k}} \leq O(\log n) \text{ !!}$$

Faster Algorithm for IP?

$$1 \le \mu(K, \mathbb{Z}^n) \min_{\substack{P \in \mathbb{Z}^{k \times n} \\ rk(P) = k \ge 1}} \mathrm{vol}_k(PK)^{\frac{1}{k}} \le O(\log n)$$

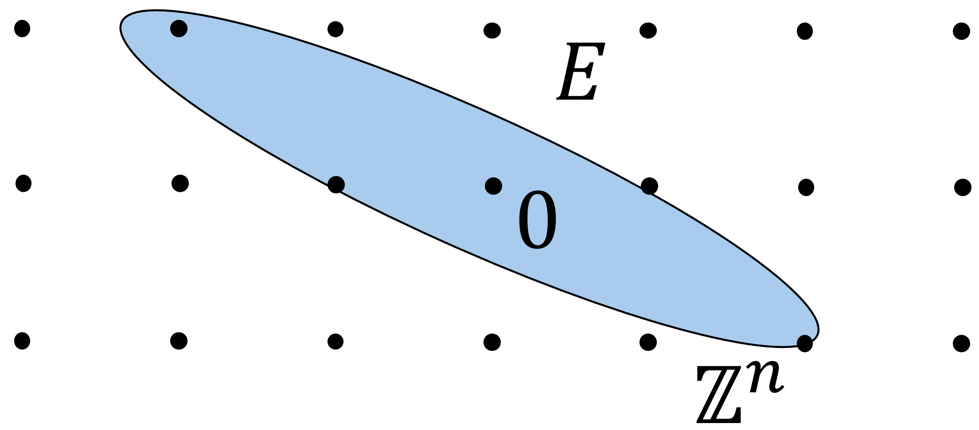
D. `12: Assuming KL conjecture

+ P computable in $(\log n)^{O(n)}$ time

then there is $(\log n)^{O(n)}$ time algorithm for IP.

ℓ_2 Kannan-Lovász Conjecture

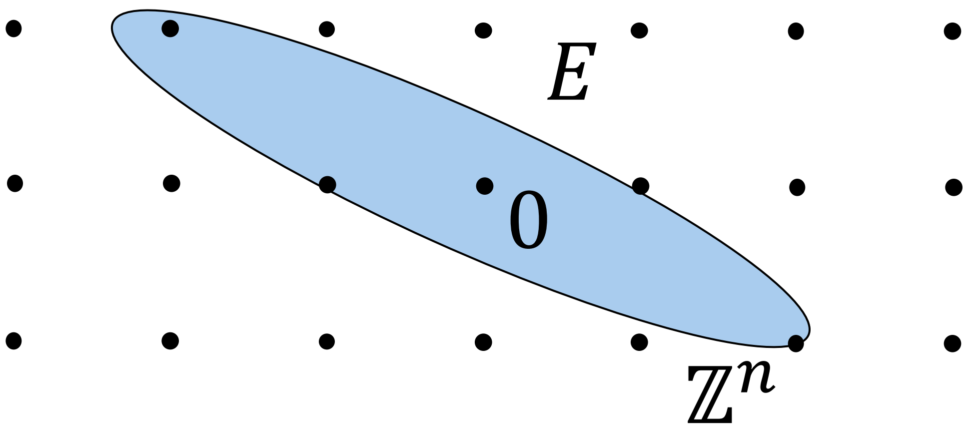
Does the conjecture hold for ellipsoids?



An ellipsoid is $E = TB_2^n$

ℓ_2 Kannan-Lovász Conjecture

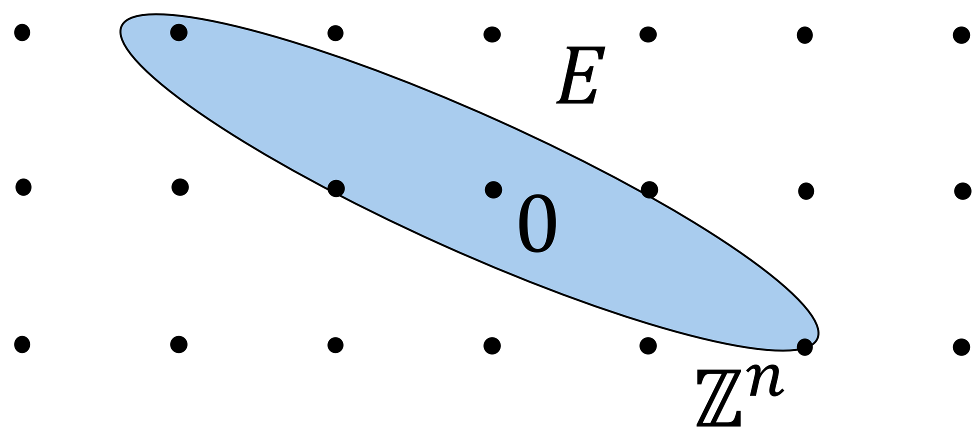
Answer: YES* [Regev-S.Davidowitz 17]



* up to polylogarithmic factors

ℓ_2 Kannan-Lovász Conjecture

Can we compute the projection P?



THIS TALK: YES, in $2^{O(n)}$ time.

Outline

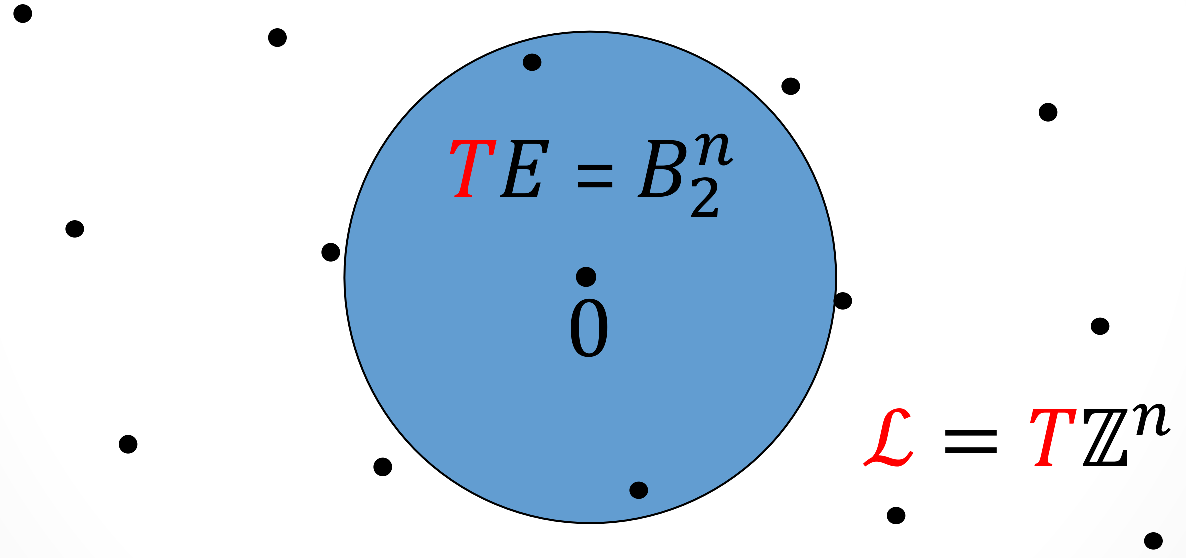
1. Integer Programming and
   the Kannan-Lovász (KL) Conjecture.

2. ℓ_2 KL Conjecture &
   the Reverse Minkowski Conjecture.

3. Finding dense lattice subspaces.

ℓ_2 Kannan-Lovász Conjecture

Easier to think of Euclidean ball vs general lattice.

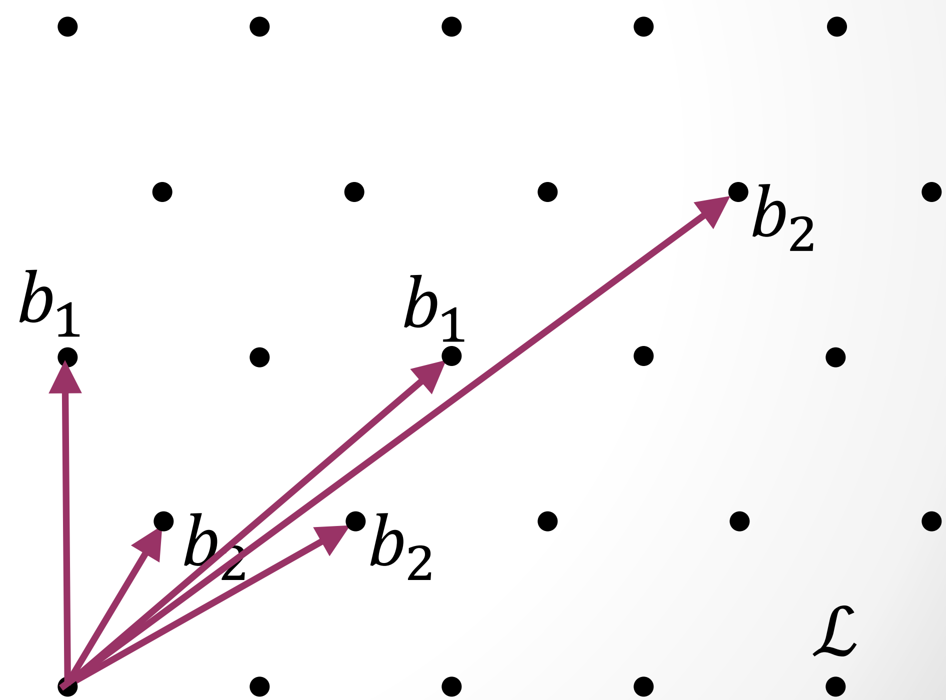


Lattices

A lattice $\mathcal{L} \subseteq \mathbb{R}^n$ is $B\mathbb{Z}^n$ for a basis $B = (b_1, \dots, b_n)$.

$\mathcal{L}(B)$ denotes the lattice generated by B.

Note: a lattice has many equivalent bases.

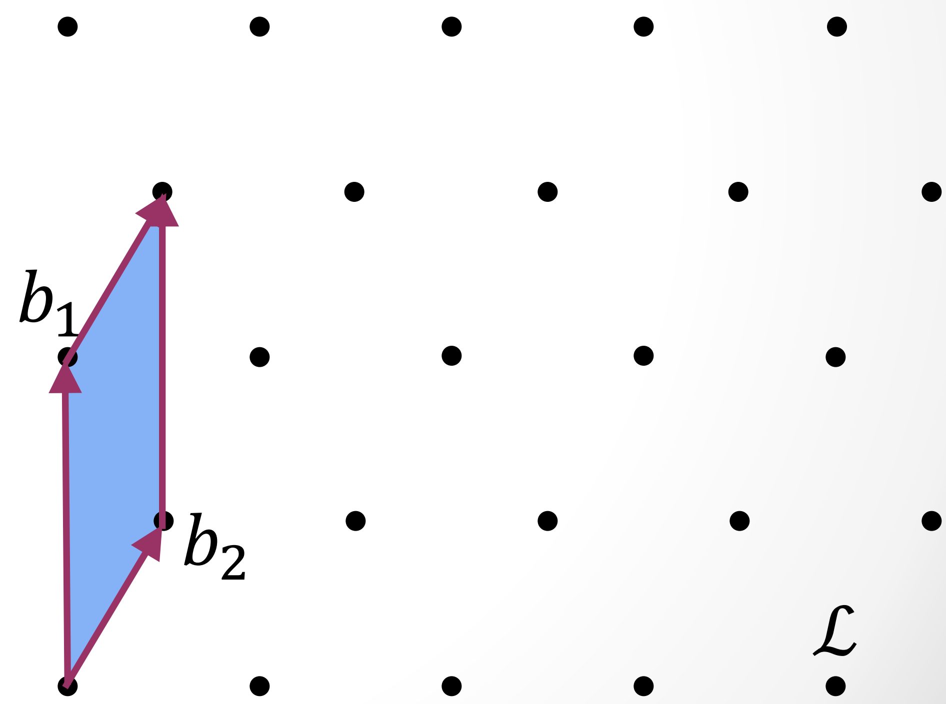


Lattices

A lattice $\mathcal{L} \subseteq \mathbb{R}^n$ is $B\mathbb{Z}^n$ for a basis $B = (b_1, \dots, b_n)$.

$\mathcal{L}(B)$ denotes the lattice generated by B.

The determinant of $\mathcal{L}$ is $|\det(B)|$.

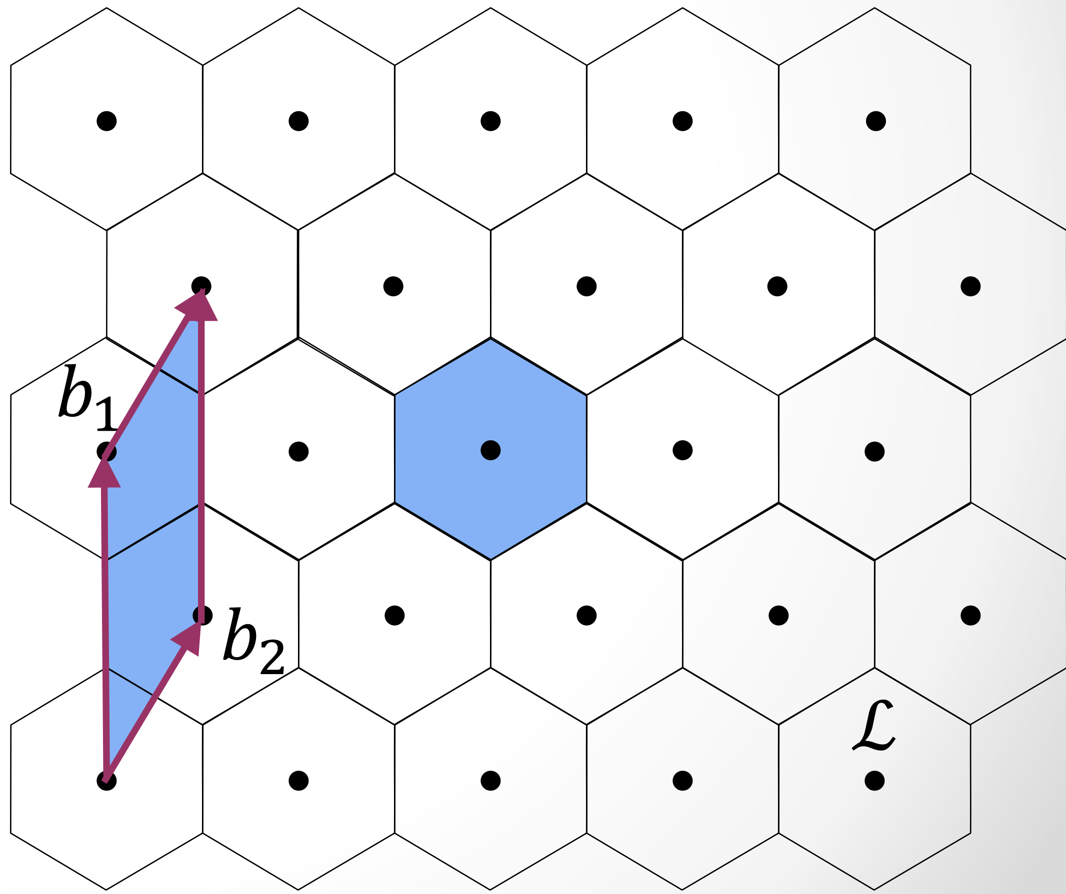


Lattices

A lattice $\mathcal{L} \subseteq \mathbb{R}^n$ is $B\mathbb{Z}^n$ for a basis $B = (b_1, \ldots, b_n)$.

$\mathcal{L}(B)$ denotes the lattice generated by B.

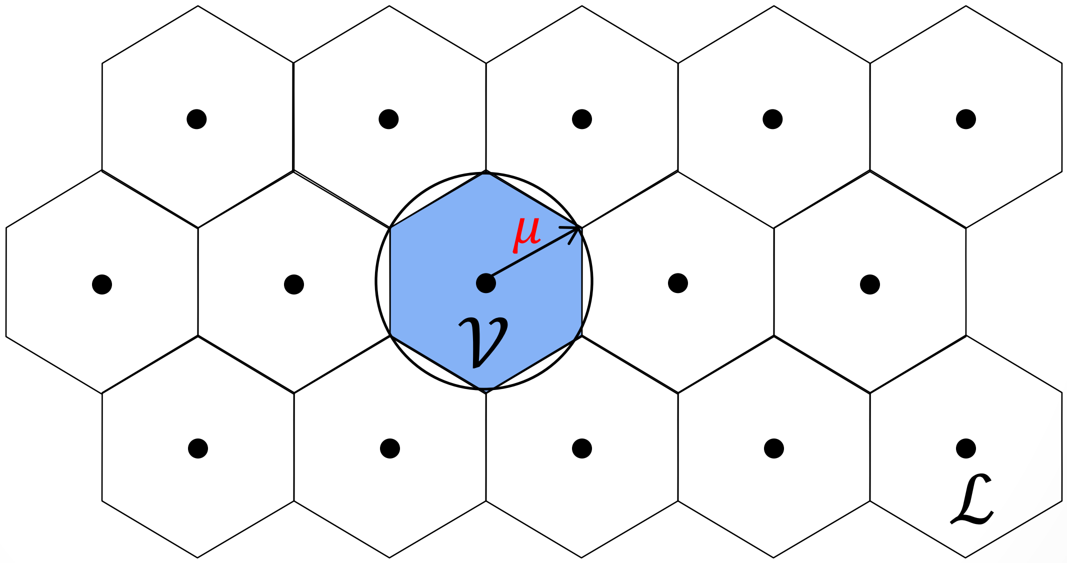
The determinant of $\mathcal{L}$ is $|\det(B)|$.
Equal to volume of any **tiling** set.



ℓ_2 Covering Radius

$$\mu(\mathcal{L}) := \mu(B_2^n, \mathcal{L})$$

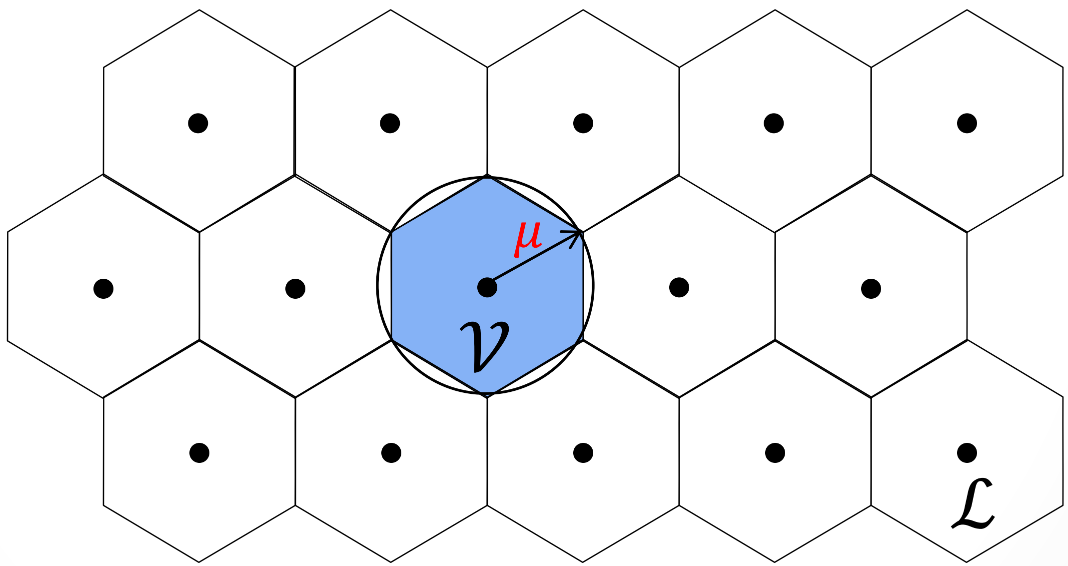
Distance of farthest point to the lattice $\mathcal{L}$.



Voronoi cell $\mathcal{V}$:= all points closer to 0

Volumetric Lower Bounds

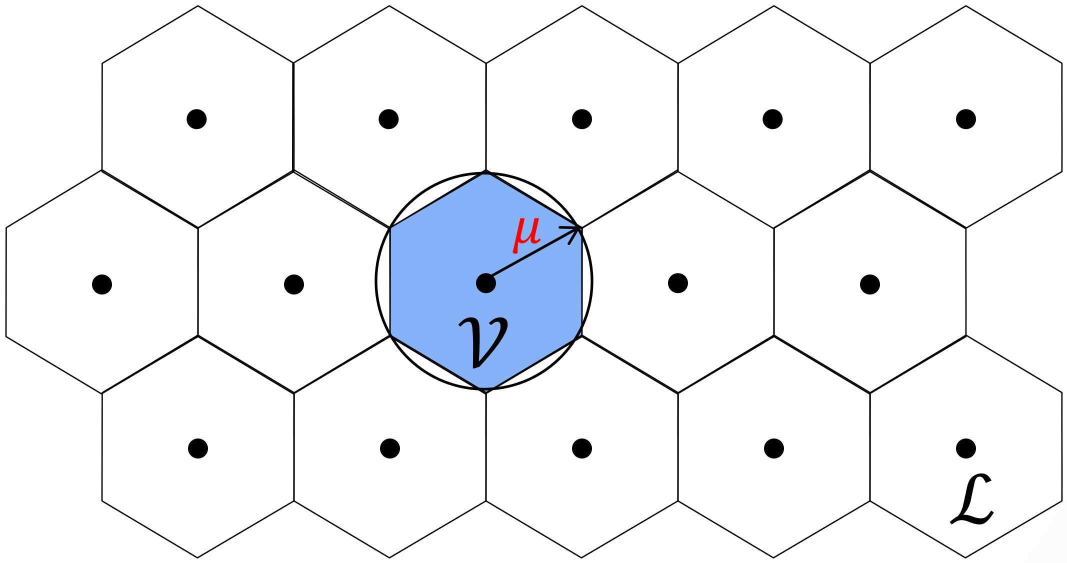
$$\mathrm{vol}_n\big(B_2^n \mu(\mathcal{L})\big) \geq \mathrm{vol}_{\mathrm{n}}(\mathcal{V}) = \det(\mathcal{L})$$



Voronoi cell $\mathcal{V}$:= all points closer to 0

Volumetric Lower Bounds

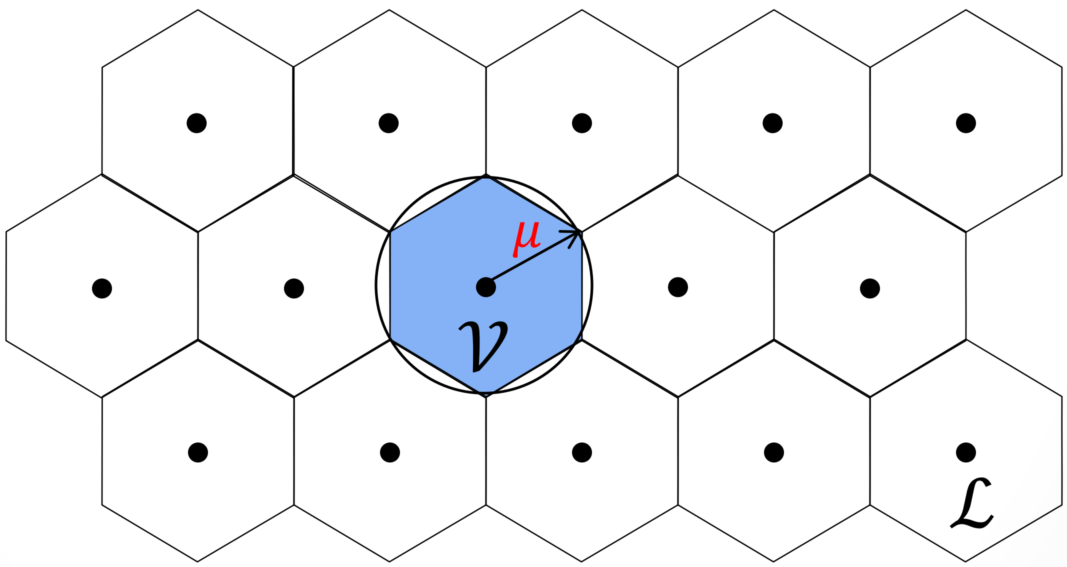
$$\mu(\mathcal{L}) \;\geq \mathrm{vol_n}(B_2^n)^{-\frac{1}{n}} \det(\mathcal{L})^{\frac{1}{n}}$$



Voronoi cell $\mathcal{V}$:= all points closer to 0

Volumetric Lower Bounds

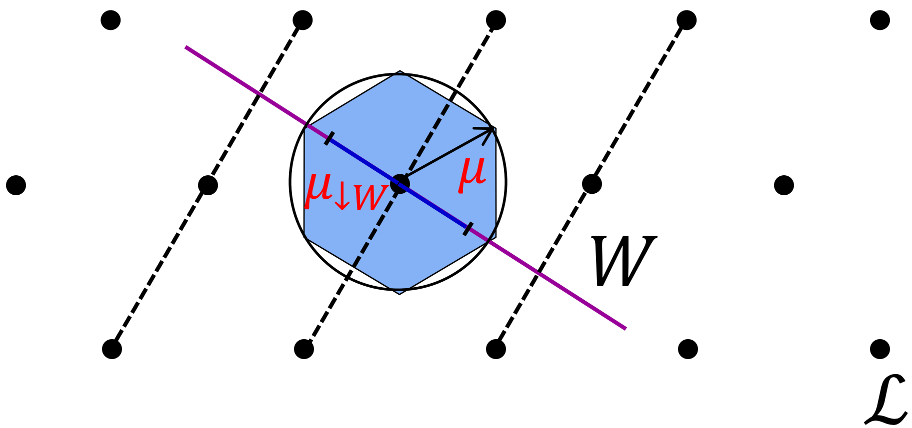
$$\mu(\mathcal{L}) \gtrsim \sqrt{n}\,\det(\mathcal{L})^{\frac{1}{n}}$$



Voronoi cell $\mathcal{V}$:= all points closer to 0

Volumetric Lower Bounds

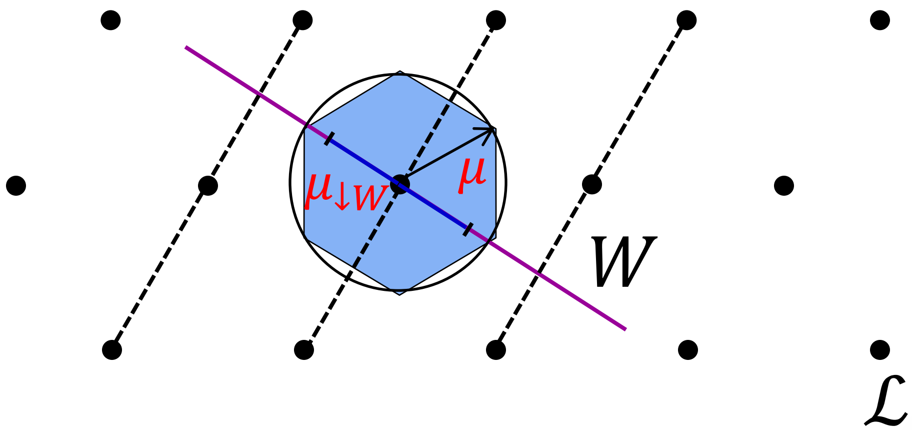
$$\mu(\mathcal{L}) \geq \mu(\mathcal{L}_{\downarrow W})$$



$\mathcal{L}_{\downarrow W}$ projection onto W

Volumetric Lower Bounds

$$\mu(\mathcal{L}) \geq \mu(\mathcal{L}_{\downarrow W}) \gtrsim \sqrt{k}\,\det(\mathcal{L}_{\downarrow W})^{\frac{1}{k}}$$

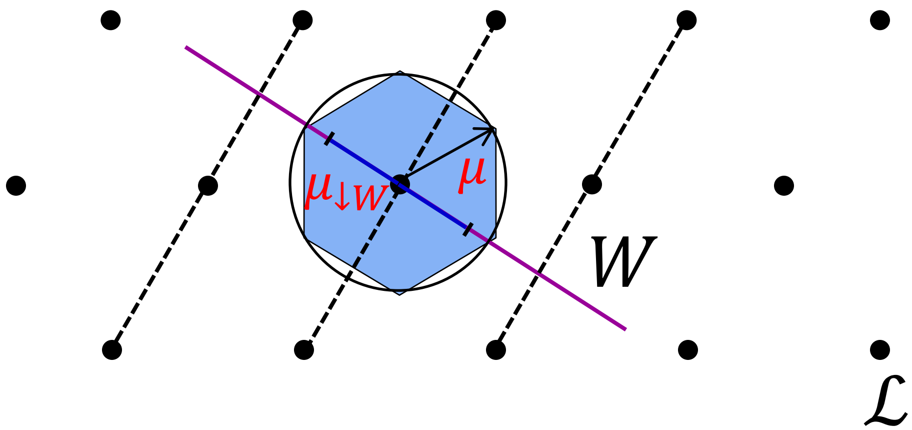


$\mathcal{L}_{\downarrow W}$ projection onto W

$\dim(W) = k \geq 1$

Volumetric Lower Bounds

$$\mu(\mathcal{L}) \gtrsim \max_{\dim(W)=k\geq 1} \sqrt{k}\,\det(\mathcal{L}_{\downarrow W})^{\frac{1}{k}}$$



$\mathcal{L}_{\downarrow W}$ projection onto W

$\dim(W) = k \geq 1$

ℓ_2 Kannan-Lovász Conjecture

Define $C_{KL,2}(n)$ to be smallest number such that

$$\mu(\mathcal{L}) \leq C_{KL,2}(n) \max_{\dim(W)=k\geq 1} \sqrt{k} \det(\mathcal{L}_{\downarrow W})^{\frac{1}{k}}$$

for all lattices of dimension at most n.

$C_{KL,2}(n) = \Omega(\sqrt{\log n})$

Lower bound for $\mathcal{L}$ with basis $e_1, \frac{1}{\sqrt{2}} e_2, \ldots, \frac{1}{\sqrt{n}} e_n$.

KL Bounds

$$\mu(\mathcal{L}) \leq C_{KL,2}(n) \max_{\dim(W)=k\geq 1} \sqrt{k} \det(\mathcal{L}_{\downarrow W})^{\frac{1}{k}}$$

Kannan-Lovász `88: $\sqrt{n}$

D. Regev `16: $\log^{O(1)} n$

Assuming Reverse Minkowski Conjecture.

Regev, S.Davidowitz `17: $\log^{3/2} n$

Reverse Minkowski Conjecture is proved!

Our Results

n dimensional lattice $\mathcal{L} \coloneqq \mathcal{L}(B)$

1. Can compute subspace W, $\dim(W) = k \geq 1$

$$\mu(\mathcal{L}) \leq O(\log^{2.5} n)\,\sqrt{k}\,\det(\mathcal{L}_{\downarrow W})^{\frac{1}{k}}$$

in $2^{O(n)}$ time with high probability.

Prior work:

Kannan Lovász \`88: $\sqrt{n}$ in $2^{O(n)}$ time.

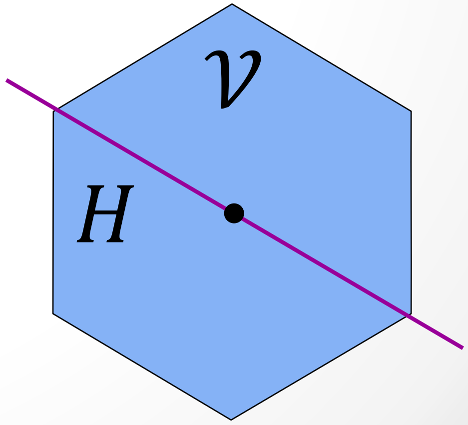
D. Micciancio \`13: best subspace in $n^{O(n^2)}$ time.

Our Results

n dimensional lattice $\mathcal{L} \coloneqq \mathcal{L}(B)$

2. Can combine lower bounds over different subspaces to certify $\mu(L)$ up to the slicing constant L_n for "stable" Voronoi cells*.

* If $\mathrm{vol_n}(\mathcal{V}) = 1$
can find hyperplane H s.t.
$\mathrm{vol_{n-1}}(\mathcal{V} \cap H) = \Omega(\frac{1}{L_n})$

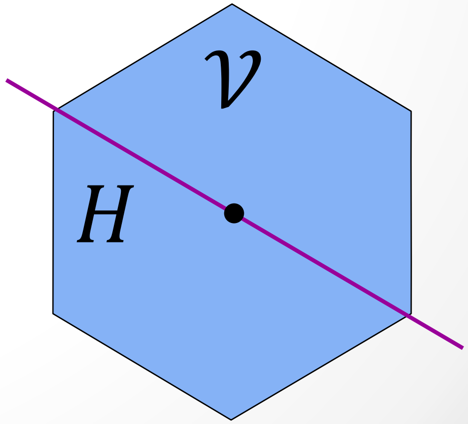


Our Results

n dimensional lattice $\mathcal{L} := \mathcal{L}(B)$

2. Can combine lower bounds over different subspaces to certify $\mu(L)$ up to the slicing constant L_n for "stable" Voronoi cells*.

Slicing Conjecture:
$L_n = O(1)$ for all convex bodies!
For "stable" Voronoi cells:
$L_n = O(\log n)$ [RS `17]



Notation

$M \subseteq \mathcal{L}$ sublattice of dimension k

Convention: $M = \{0\}$ then $\det(M) := 1$.

Normalized Determinant:

$$\mathrm{nd}(M) := \det(M)^{1/k}$$

Projected Sublattice:

$\mathcal{L}/M := \mathcal{L}$ projected onto $\mathrm{span}(M)^{\perp}$

Lower Bounds for Chains

Theorem [D. 17]:
For $\{0\} = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_k = \mathcal{L}$ then

$$\mu(\mathcal{L})^2 \gtrsim \sum_{i=1}^{k} \dim(\mathcal{L}_i/\mathcal{L}_{i-1}) \operatorname{nd}(\mathcal{L}/\mathcal{L}_{i-1})^2$$

Only "missing ingredient":
Combined with techniques from [R.S. `17] easily get tightness within slicing constant L_n.

Lower Bounds for Chains

Theorem [D. 17]:
For $\{0\} = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_k = \mathcal{L}$ then

$$\mu(\mathcal{L})^2 \gtrsim \sum_{i=1}^{k} \operatorname{dim}(\mathcal{L}_i/\mathcal{L}_{i-1}) \operatorname{nd}(\mathcal{L}/\mathcal{L}_{i-1})^2$$

Proof Idea:

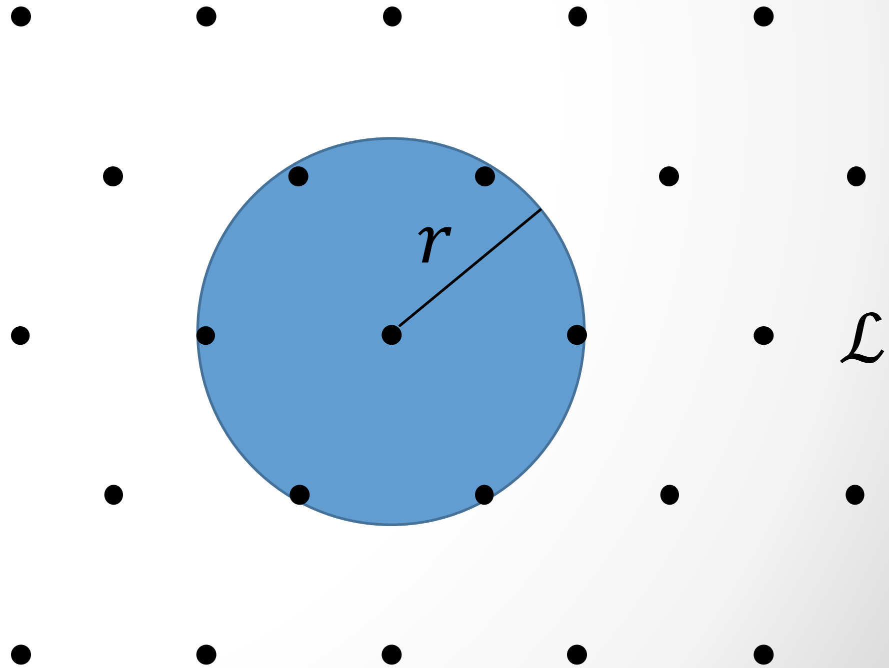
1. Establish SDP based lower bound: [D.R. `16]

$$\mu(\mathcal{L})^2 \gtrsim \max \sum_i \operatorname{rk}(P_i) \operatorname{nd}(P_i\mathcal{L})^2$$
$$\text{s.t.} \quad \sum_i P_i^* P_i \preccurlyeq I_n$$

2. Build solution to above starting from any chain.

Lattice Density

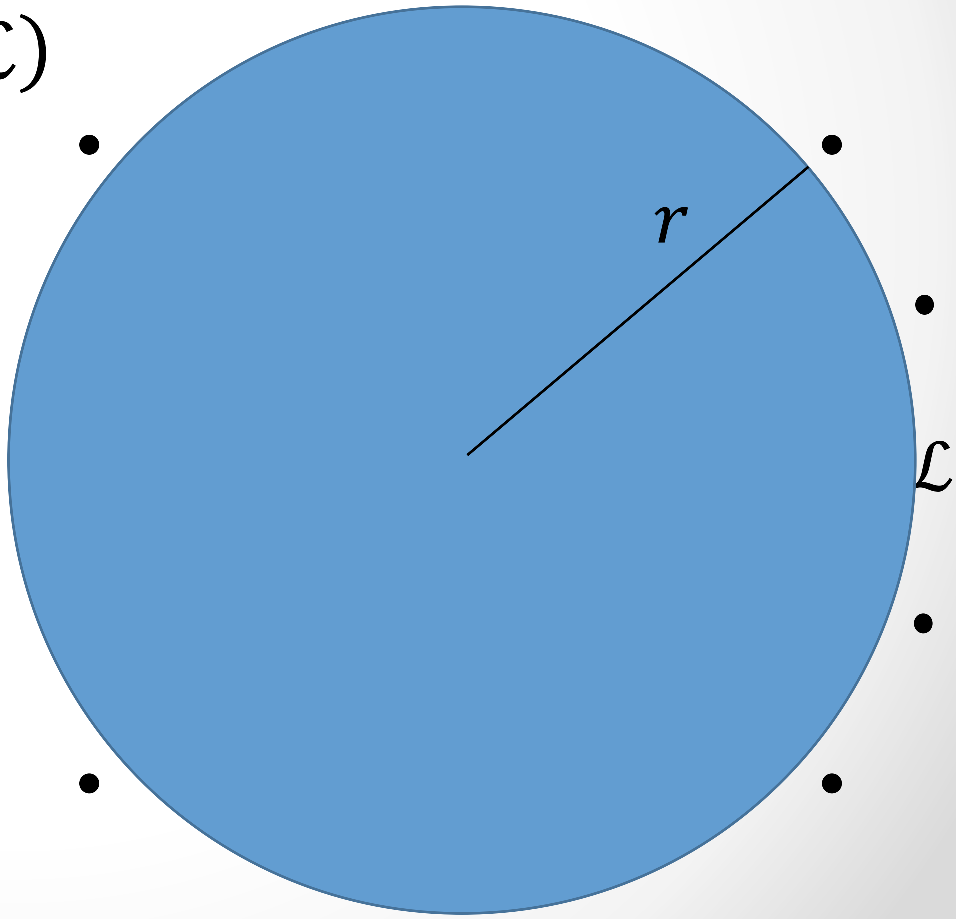
$$\lim_{r\to\infty} \frac{|\mathcal{L} \cap rB_2^n|}{\mathrm{vol}_n(rB_2^n)} = \frac{1}{\det(\mathcal{L})}$$



Lattice Density

$$\lim_{r\to\infty} \frac{|\mathcal{L} \cap rB_2^n|}{\mathrm{vol}_n(rB_2^n)} = \frac{1}{\det(\mathcal{L})}$$

Global density of lattice points per unit volume

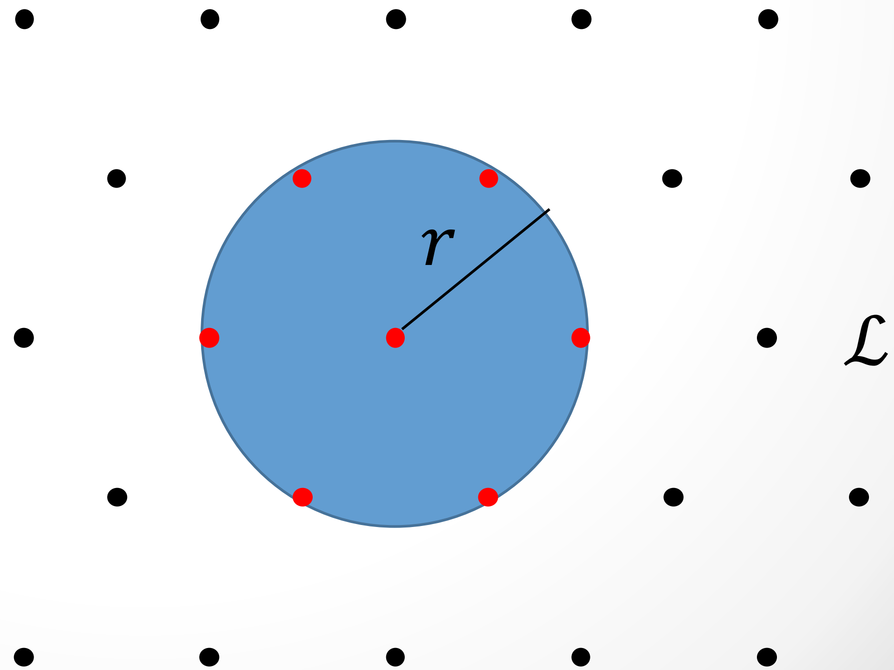


Minkowski's First Theorem

1889

Global density implies
"local density"

$$|\mathcal{L} \cap rB_2^n| \geq 2^{-n} \frac{\mathrm{vol}_n(rB_2^n)}{\det(\mathcal{L})}$$



Reverse Minkowski Theorem

Regev-S.Davidowitz `17:
$\mathcal{L}$ lattice dimension n.
If all sublattices of $\mathcal{L}$
have determinant at least 1 then:

$\mathcal{L}$ has at most $2^{O(\log^2 n \, r^2)}$ points at distance r.

Almost tight: $\mathbb{Z}^n$ has $n^{\Omega(k)}$ points at distance r
for $k \ll n$.

Outline

1. Integer Programming and the Kannan-Lovász (KL) Conjecture.

2. ℓ_2 KL Conjecture & the Reverse Minkowski Conjecture.

3. Finding dense lattice subspaces.

Notation

$M \subseteq \mathcal{L}$ sublattice of dimension k

Convention: $M = \{0\}$ then $\det(M) := 1.$

Normalized Determinant:

$$\mathrm{nd}(M) := \det(M)^{1/k}$$

Projected Sublattice:

$$\mathcal{L}/M := \mathcal{L} \text{ projected onto } \mathrm{span}(M)^{\perp}$$

Densest Subspace Problem

$$\mathrm{nd}^*(\mathcal{L}) := \min_{\substack{M \subseteq \mathcal{L} \\ M \neq \{0\}}} \mathrm{nd}(M)$$

α-DSP: Given $\mathcal{L}$ find $M \subseteq \mathcal{L}$, $M \neq \{0\}$ such that $\mathrm{nd}(M) \leq \alpha \, \mathrm{nd}^*(\mathcal{L})$.

Remark: dimension of M is not fixed!

Key primitive for finding sparse lattice projections. Will focus on this problem.

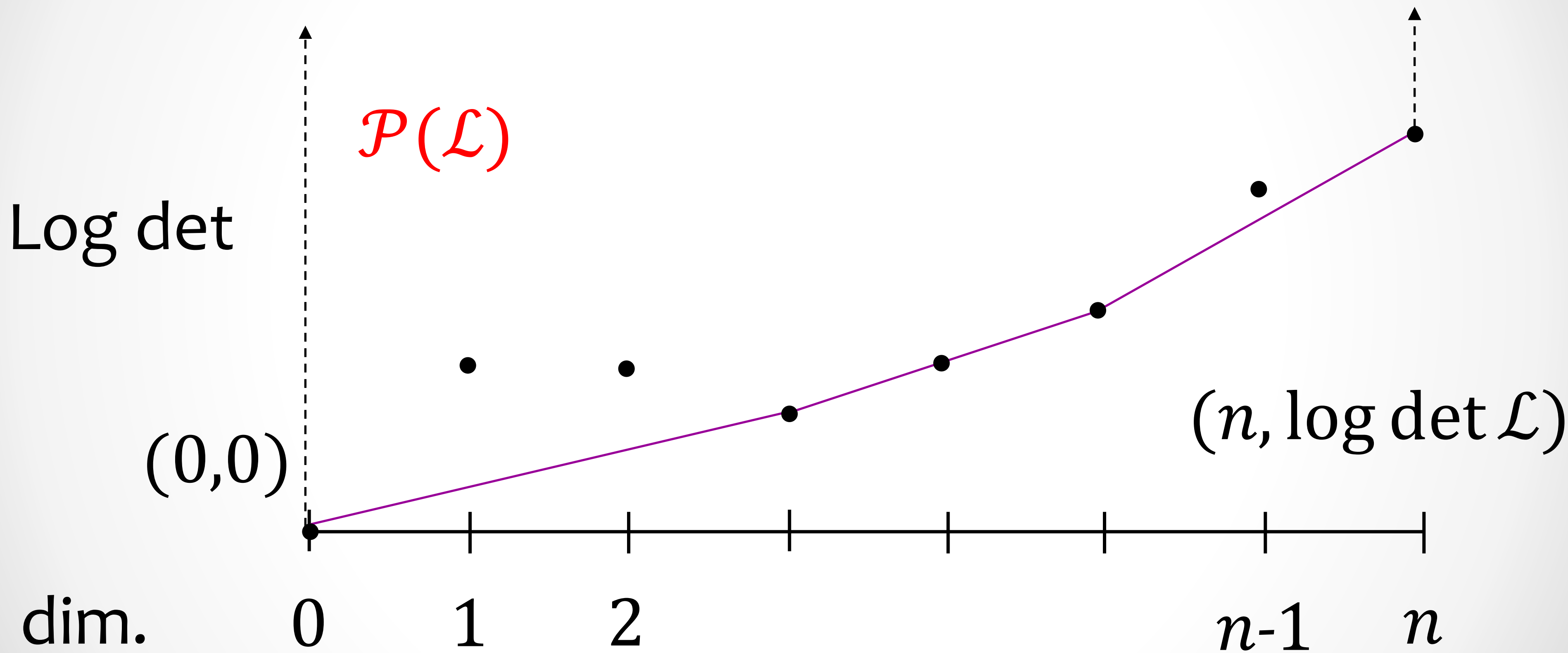
Densest Subspace Problem

Theorem:
Can solve $O(\log n)$-DSP in $2^{O(n)}$ time with high probability.

High Level Approach:
If $\mathcal{L}$ is not approximate minimizer:
find $y \neq 0$, orthogonal to actual minimizer, and recurse on $\mathcal{L} \cap y^{\perp}$

Canonical Polytope [Stuhler 76]

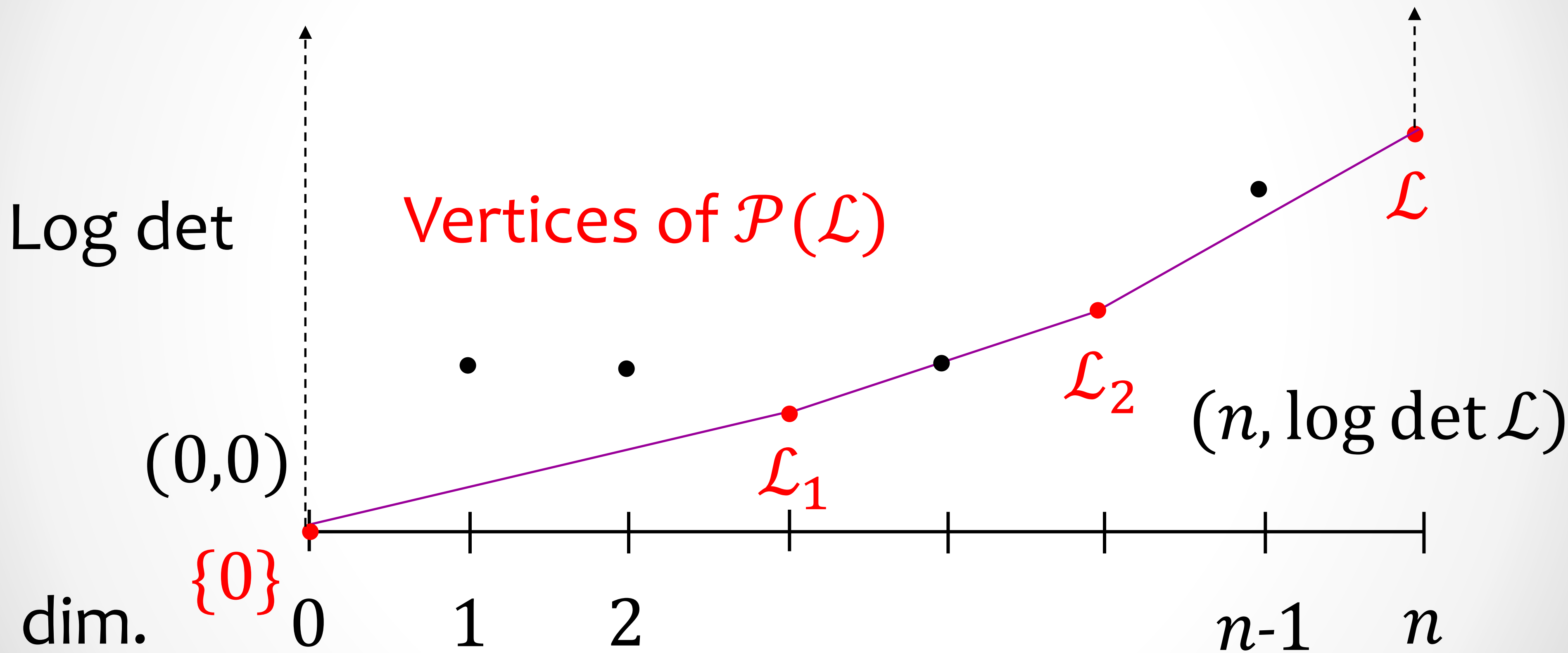
n dimensional lattice $\mathcal{L}$



$$\{(k, \log\det(M)) : \text{sublattice } M \subseteq \mathcal{L}, \dim(M) = k\}$$

Canonical Filtration [Stuhler 76]

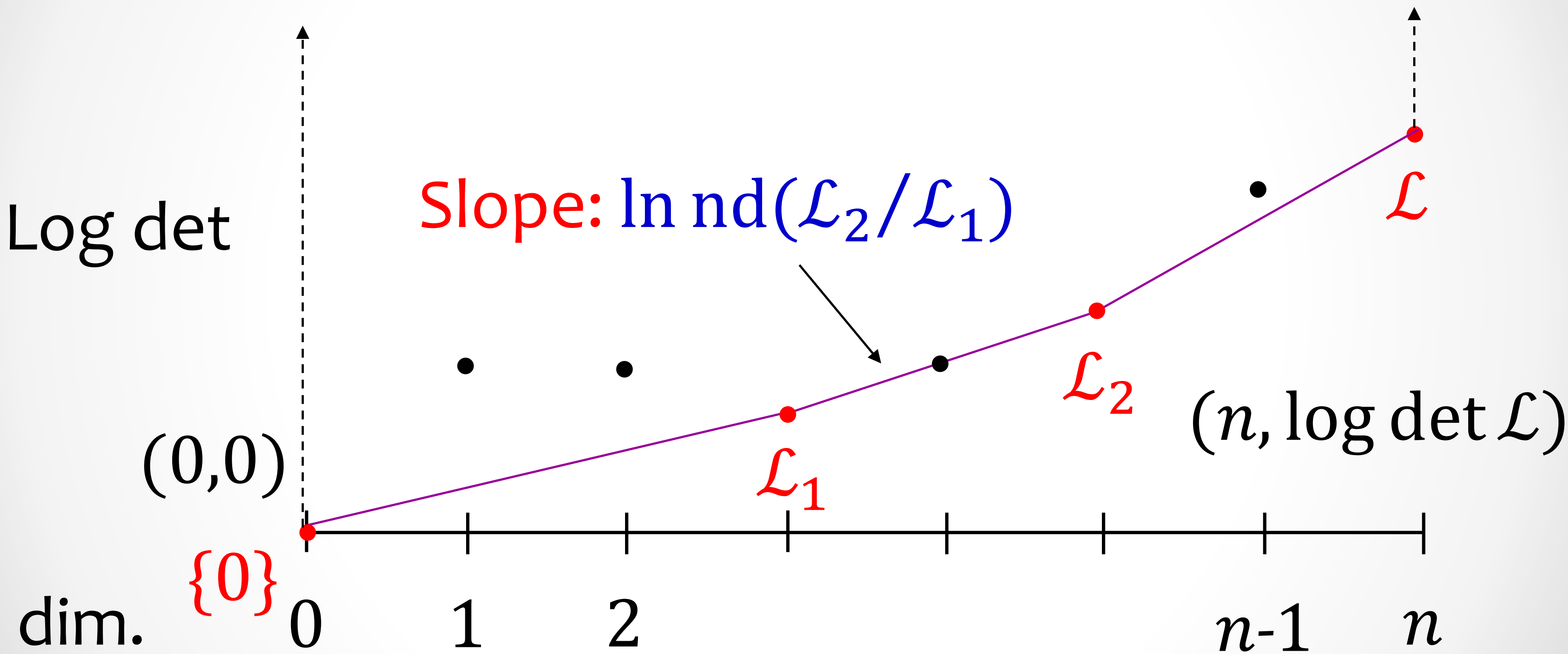
n dimensional lattice $\mathcal{L}$



Form Chain: $\{0\} = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_k = \mathcal{L}$

Canonical Filtration [Stuhler 76]

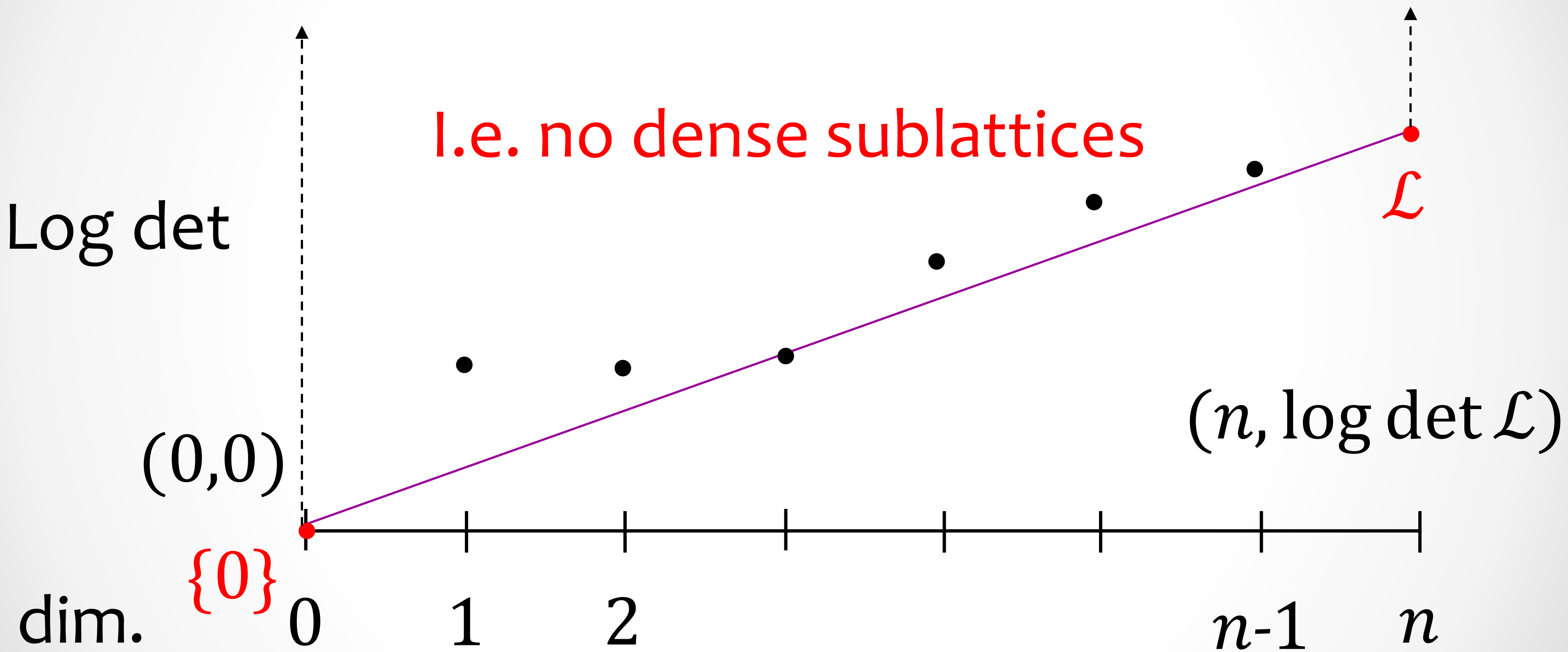
n dimensional lattice $\mathcal{L}$



Form Chain: $\{0\} = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_k = \mathcal{L}$

Stable Lattice [Stuhler 76]

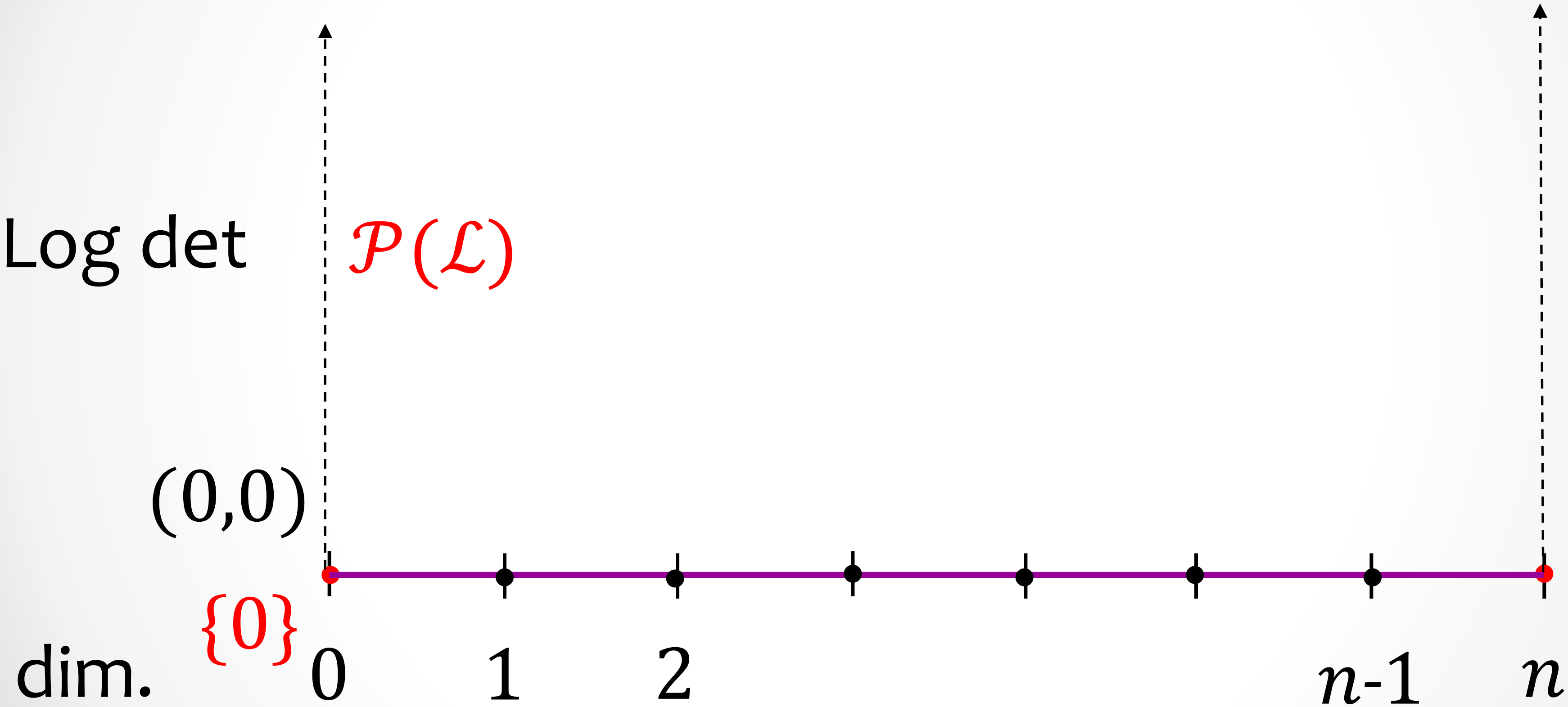
n dimensional lattice $\mathcal{L}$ is stable



If canonical filtration is trivial: $\{0\} \subset \mathcal{L}$

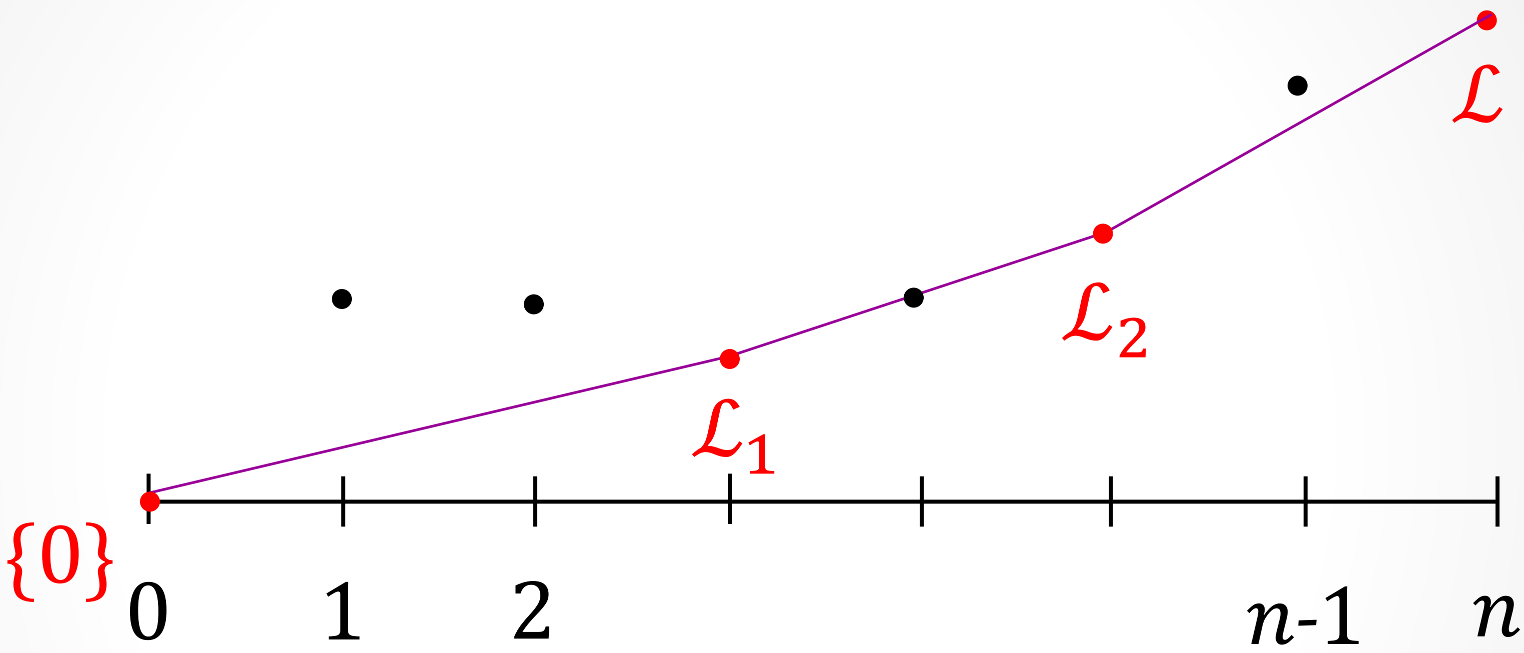
Stable Lattice [Stuhler 76]

Example: $\mathcal{L} = \mathbb{Z}^n$



$\mathbb{Z}^n$ has trivial filtration: $\{0\} \subset \mathbb{Z}^n$

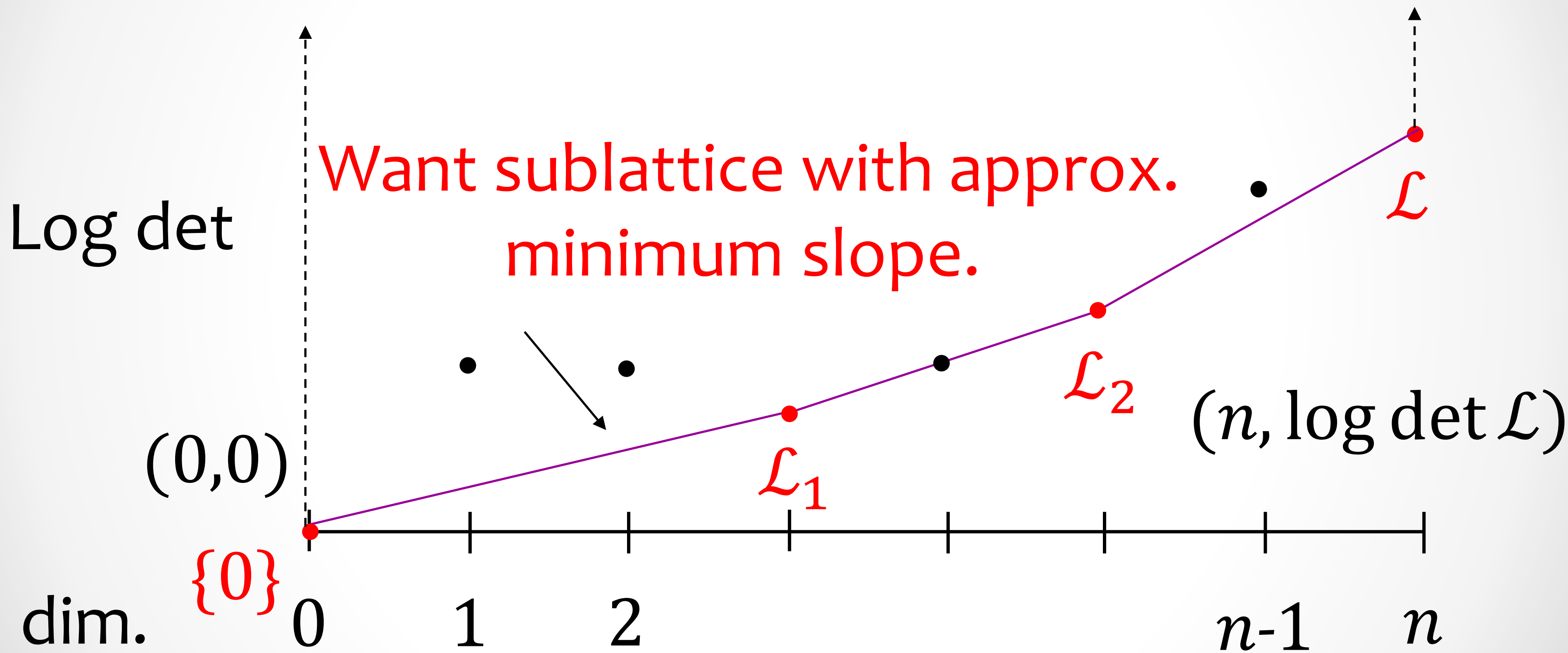
Canonical Filtration [Stuhler 76]



1. Form Chain: $\{0\} = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_k = \mathcal{L}$.
2. Blocks $\mathcal{L}_i/\mathcal{L}_{i-1}$ are stable.
3. Slope increasing: $\mathrm{nd}(\mathcal{L}_i/\mathcal{L}_{i-1}) < \mathrm{nd}(\mathcal{L}_{i+1}/\mathcal{L}_i)$.

Densest Subspace Problem

n dimensional lattice $\mathcal{L}$



Densest Subspace Problem

High Level Approach:
If $\mathcal{L}$ is not approximate minimizer:
find $y \neq 0$, orthogonal to actual minimizer,
and recurse on $\mathcal{L} \cap y^{\perp}$

Q: Where to find y?
A: The dual lattice $\mathcal{L}^*$

Q: How to find it in $\mathcal{L}^*$?
A: Discrete Gaussian sampling

Dual Lattice

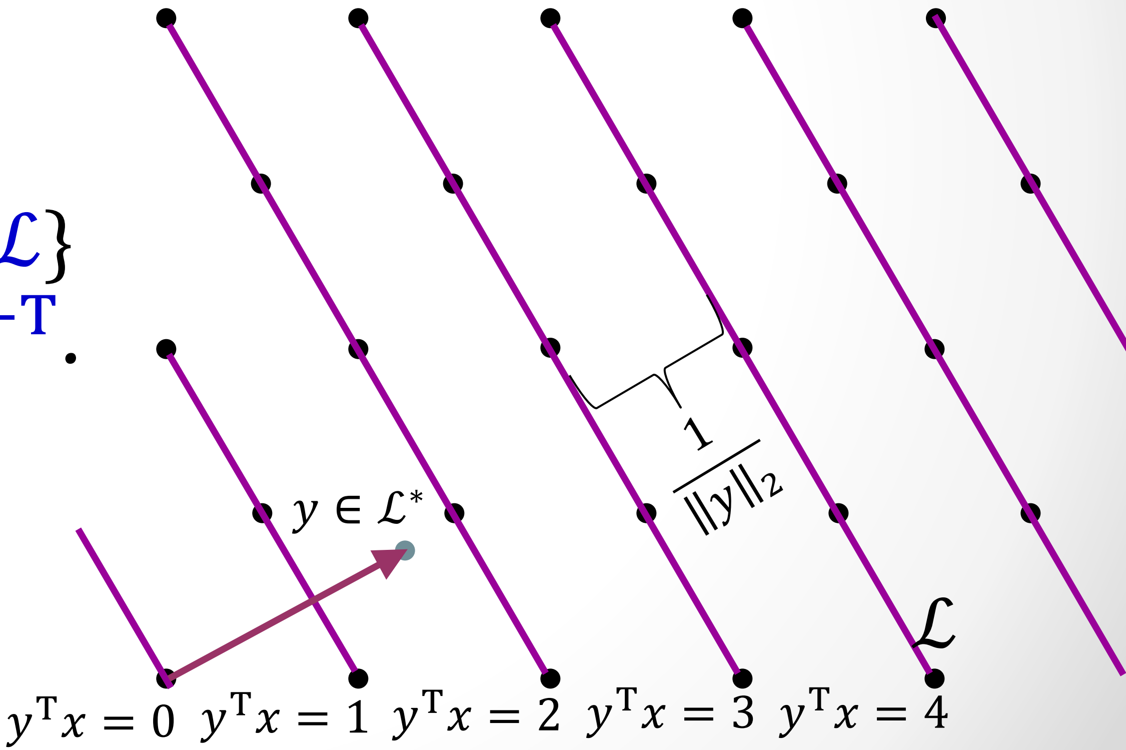
A lattice $\mathcal{L} \subseteq \mathbb{R}^n$ is $B\mathbb{Z}^n$ for a basis $B = (b_1, \ldots, b_n)$.

The dual lattice is

$$\mathcal{L}^* = \{y \in \mathrm{span}(\mathcal{L}):\ y^{\mathrm{T}} x \in \mathbb{Z}\ \ \forall x \in \mathcal{L}\}$$

$\mathcal{L}^*$ is generated by $B^{-\mathrm{T}}$.

Remark: $(\mathbb{Z}^n)^* = \mathbb{Z}^n$

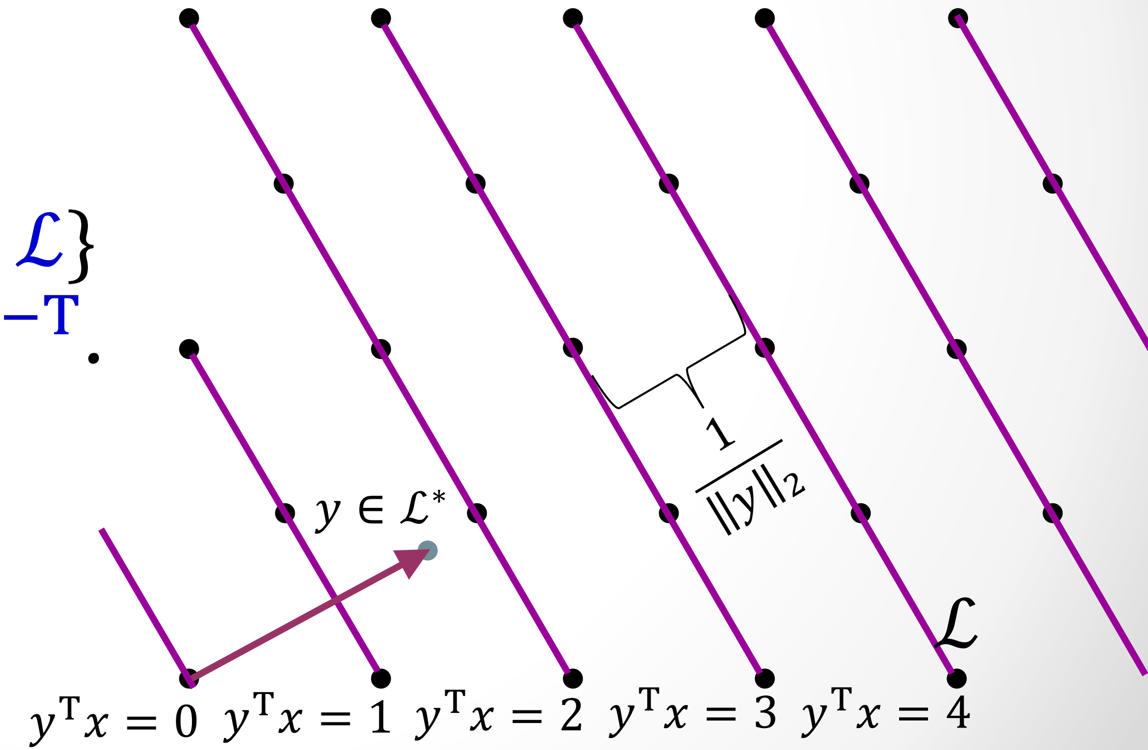


Dual Lattice

A lattice $\mathcal{L} \subseteq \mathbb{R}^n$ is $B\mathbb{Z}^n$ for a basis $B = (b_1, \dots, b_n)$.

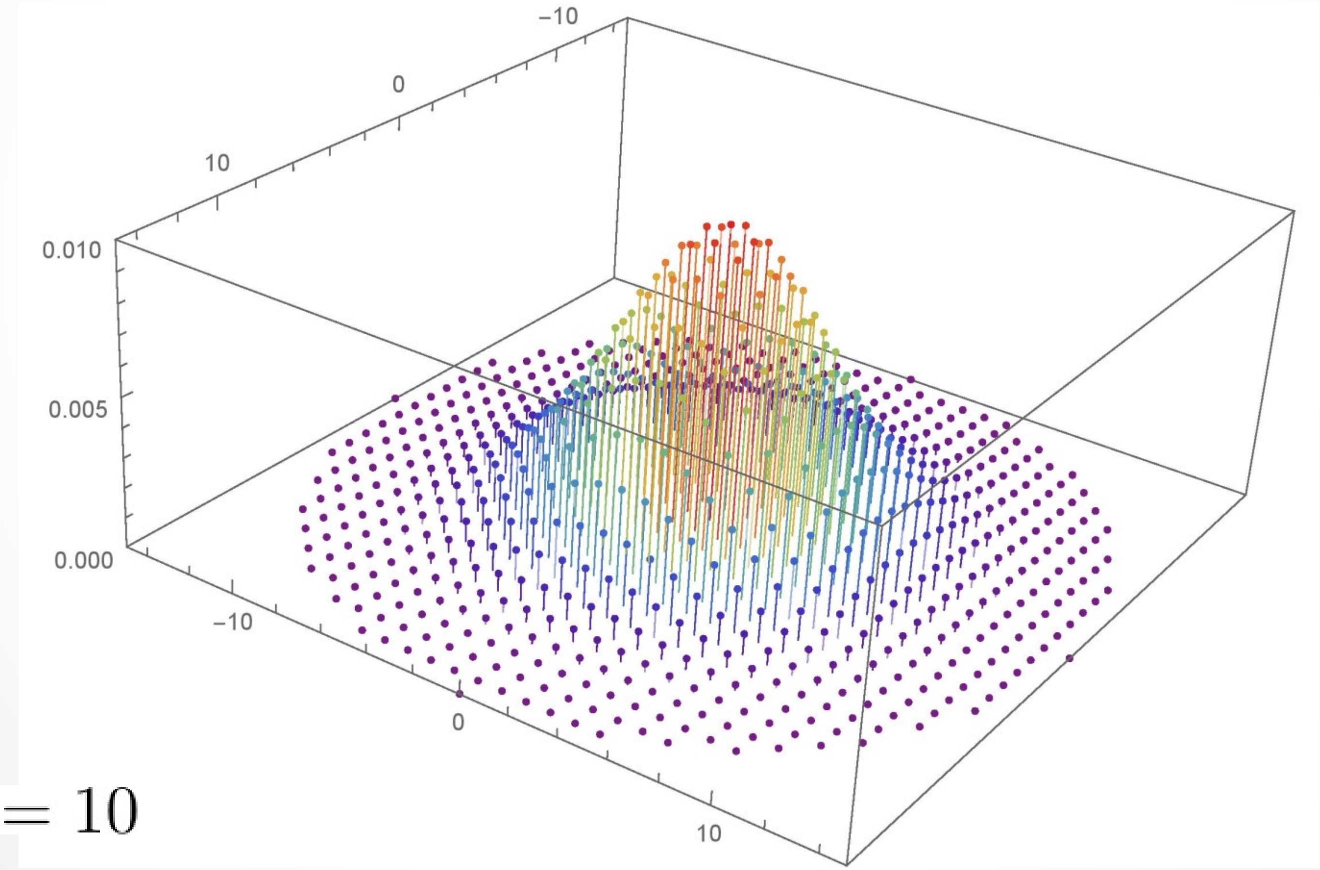
The dual lattice is
$\mathcal{L}^* = \{y \in \mathrm{span}(\mathcal{L}):$
$\quad y^{\mathrm{T}} x \in \mathbb{Z} \;\; \forall x \in \mathcal{L}\}$
$\mathcal{L}^*$ is generated by $B^{-\mathrm{T}}$.

$\det(\mathcal{L}^*) = 1/\det(\mathcal{L})$



Discrete Gaussian Distribution

$$D_{\mathcal{L},s} := \Pr[\mathbf{y}] \propto e^{-\|\mathbf{y}\|^2/s^2}$$



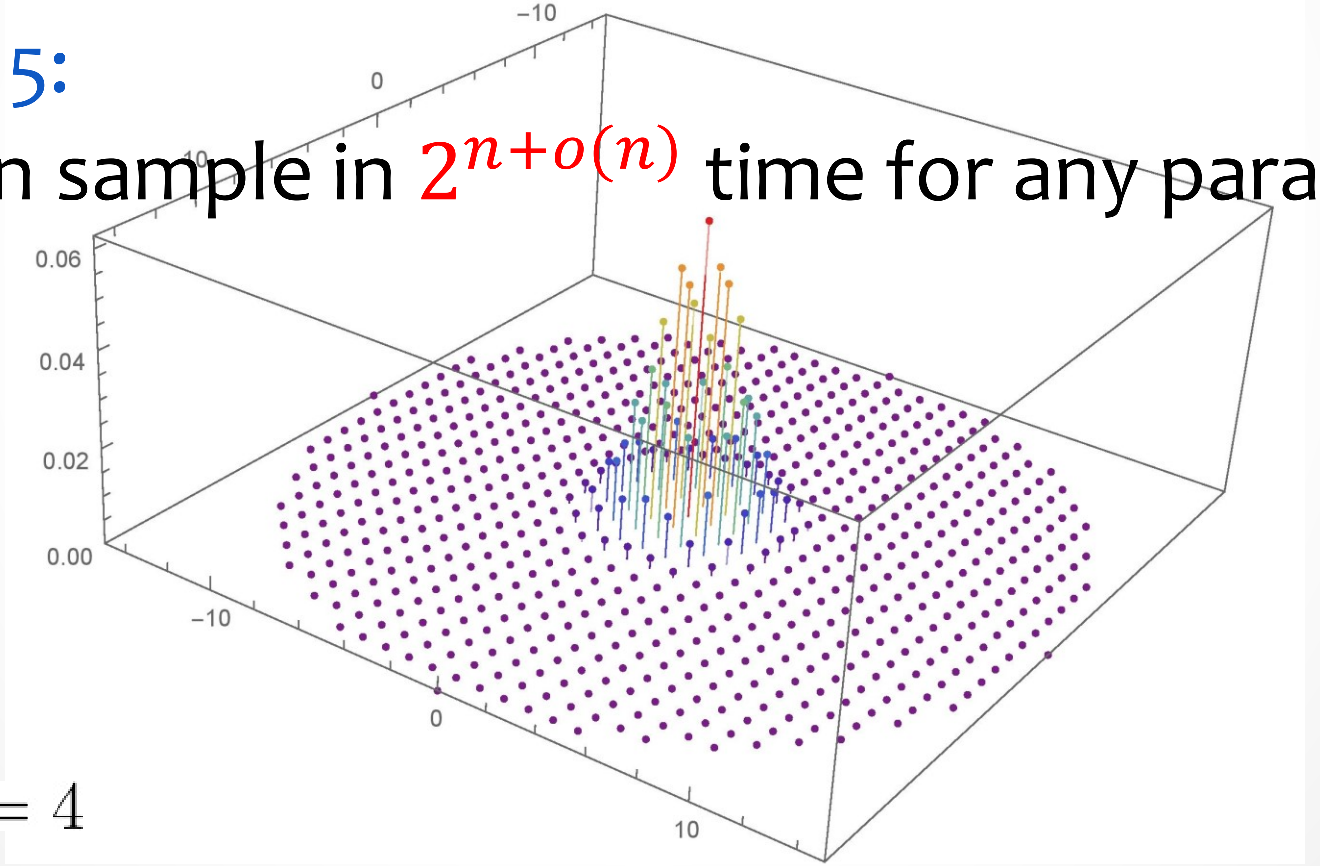
$s = 10$

Discrete Gaussian Distribution

$$D_{\mathcal{L},s} := \Pr[\mathbf{y}] \propto e^{-\|\mathbf{y}\|^2/s^2}$$

ADRS `15:

Can sample in $2^{n+o(n)}$ time for any parameter.



$s = 4$

Main Procedure

Repeat until $\mathcal{L} = \{0\}$

$s \leftarrow \mathrm{nd}(\mathcal{L})$

Update $M \leftarrow \mathcal{L}$ if $\mathrm{nd}(M) > s$

Sample $y \sim D_{\mathcal{L}^*, c/s}$ until $y \neq 0$

$\mathcal{L} \leftarrow \mathcal{L} \cap y^{\perp}$

Main Lemma: At any iteration,
if $\mathcal{L}$ not $O(\log n)$ approximate minimizer,
then $\mathcal{L} \cap y^{\perp}$ contains minimizer w.p. $\Omega(1)$.

Proc. finds apx minimizer with prob. $2^{-O(n)}$.

Proof of Main Lemma

wlog $\det(\mathcal{L}) = \det(\mathcal{L}^*) = 1$
$\mathcal{L}_1 \subseteq \mathcal{L}$ densest sublattice

sample $y \sim D_{\mathcal{L}^*,c}$

If $\mathrm{nd}(\mathcal{L}_1) \ll \frac{1}{\log n}$

must show that $y \neq 0$ and $y \perp \mathcal{L}_1$ w.p. $\Omega(1)$.

$\det(\mathcal{L}) = \det(\mathcal{L}^*) = 1$

$\mathcal{L}_1 \subseteq \mathcal{L}$ densest sublattice, $\mathrm{nd}(\mathcal{L}_1) \ll \frac{1}{\log n}$

Sample $y \sim D_{\mathcal{L}^*,c}$

Want:

1. $\Pr[y = 0] \leq \epsilon$
2. $\Pr[y \in \mathcal{L}^* \cap \mathcal{L}_1^\perp] \geq 1 - \epsilon$

$$\det(\mathcal{L}) = \det(\mathcal{L}^*) = 1$$

1. $$\Pr_{y \sim D_{\mathcal{L}^*,c}}[y = 0] = \frac{1}{\rho_c(\mathcal{L}^*)}$$

$$\leq \frac{1}{|\mathcal{L}^* \cap \sqrt{n} B_2^n| e^{-n/c^2}}$$

(By Minkowski) $$\leq \frac{1}{2^n e^{-n/c^2}} = o(1)$$

$$\rho_c(A) := \sum_{x \in A} e^{-\|x/c\|^2}$$

$\det(\mathcal{L}) = \det(\mathcal{L}^*) = 1$

$\mathcal{L}_1 \subseteq \mathcal{L}$ densest sublattice, $\mathrm{nd}(\mathcal{L}_1) \ll 1/\log n$

$W := \mathcal{L}_1^{\perp}$

2. $\Pr_{y \sim D_{\mathcal{L}^*,c}} [y \in W] = \dfrac{\rho_c(\mathcal{L}^* \cap W)}{\rho_c(\mathcal{L}^*)}$

(ortho. is worst-case) $\geq \dfrac{\rho_c(\mathcal{L}^* \cap W)}{\rho_c(\mathcal{L}^* \cap W)\rho_c(\mathcal{L}^*/W)}$

$= \dfrac{1}{\rho_c(\mathcal{L}^*/W)}$

$\rho_c(A) := \sum_{x \in A} e^{-\|x/c\|^2}$

$\det(\mathcal{L}) = \det(\mathcal{L}^*) = 1$

$\mathcal{L}_1 \subseteq \mathcal{L}$ densest sublattice, $\mathrm{nd}(\mathcal{L}_1) \ll 1/\log n$

$W := \mathcal{L}_1^{\perp}$

2. Need to show $\rho_c(\mathcal{L}^*/W) \leq 1 + o(1)$

Key: $\mathrm{nd}^*(\mathcal{L}^*/W) = 1/\mathrm{nd}(\mathcal{L}_1) \gg \log n$

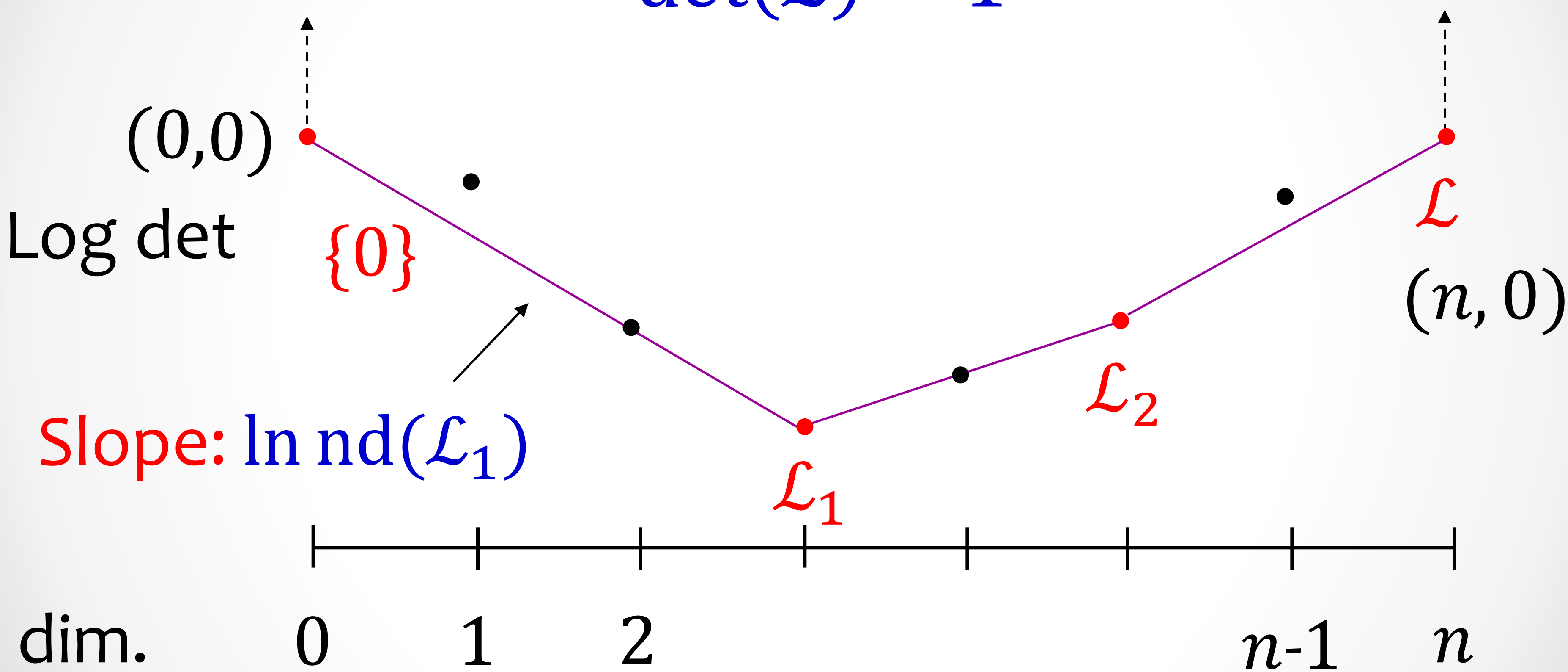
Reverse-Minkowski $\Rightarrow$

$$|(\mathcal{L}^*/W) \cap rB_2^n| \ll e^{o(r^2)}, \forall r \geq 0$$

$$\rho_c(A) := \textstyle\sum_{x \in A} e^{-\|x/c\|^2}$$

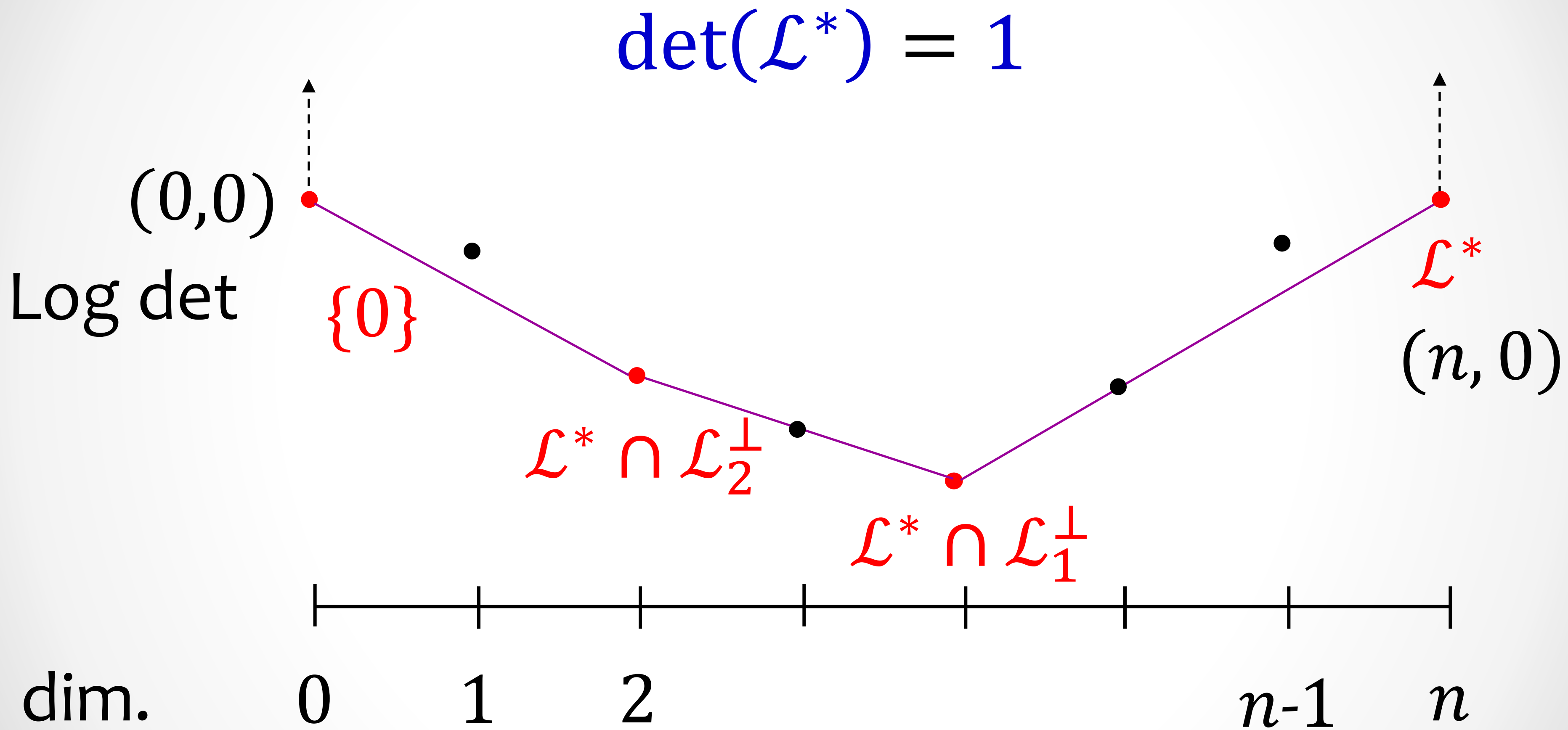
Canonical Polytope of $\mathcal{L}^*$?

$\det(\mathcal{L}) = 1$



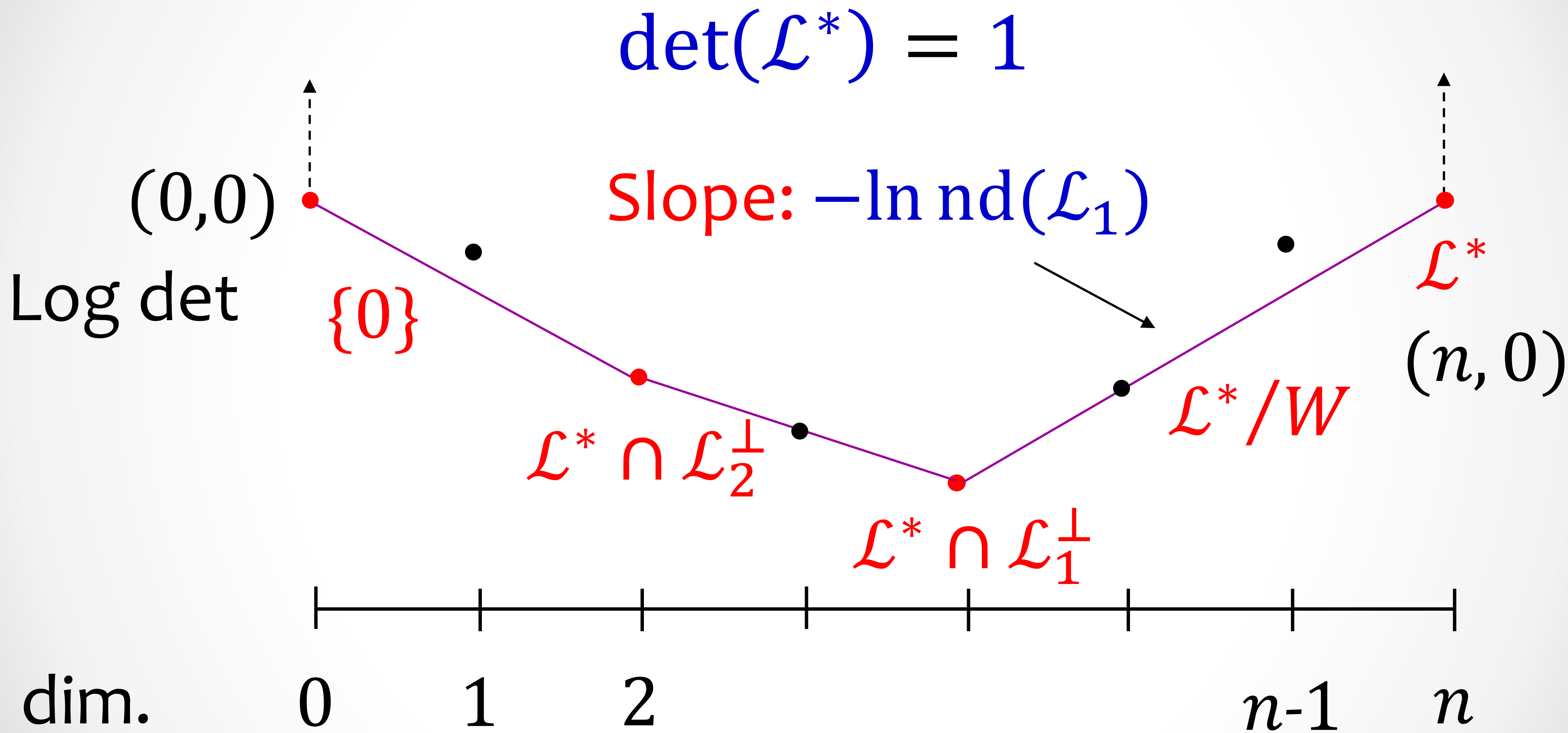
Assumption: $\det(\mathcal{L}) = 1$

Canonical Polytope of $\mathcal{L}^*$?



Map $M \to \mathcal{L}^* \cap M^\perp$

Canonical Polytope of $\mathcal{L}^*$?



$\mathcal{P}(\mathcal{L}^*)$ is "reflection" of $\mathcal{P}(\mathcal{L})$

Conclusions

1. Algorithmic version of ℓ_2 Kannan-Lovász conjecture via discrete Gaussian sampling.
2. Lower bound certificates for covering radius that are tight within $O(1)$ under slicing conjecture.

Open Problem

1. Prove KL conjecture for general convex bodies.
2. Prove Slicing conjecture for Voronoi cells.